THE HEIGHT OF COMPACT NONSINGULAR HEISENBERG-LIKE NILMANIFOLDS

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Abstract

This thesis examines the spectral ζ-function $\zeta((M,g),s)$ and the height $\zeta'((M,g),0)$ of compact nonsingular Heisenberg-like nilmanifolds (M,g). Heisenberg-like nilmanifolds were introduced by Gornet and Mast as a natural generalisation of nilmanifolds of Heisenberg-type. In case M is a Heisenberg manifold $\Gamma\backslash H_n$, every metric induced by a left-invariant metric on H_n is Heisenberg-like.

The first result of the main chapter consists of formulas for the spectral ζ-function and the height which are valid for all Heisenberg-like metrics. Hereafter, by means of these formulas several results concerning the (non-)existence of global lower bounds for the height $\zeta'(\cdot,0)$ and the ζ-function $\zeta(\cdot,s)$ for positive s not in the pole set are proved. The first of these results is the existence of lower bounds for the height and the ζ-function on the moduli space of metrics of Heisenberg-type and with volume 1 on a given nilmanifold M. Next, conditions for the existence of lower bounds for the height and the ζ-function on the moduli space of Heisenberg-like metrics with volume 1 on M are investigated. In case of the height, lower bounds exist if the dimension of M equals 3 modulo 4. In case of $\zeta(\cdot,s)$, the existence of lower bounds depends on the position of s.

To understand the remaining dimensions resp. positions of s, we introduce a path in the moduli space of Heisenberg-like metrics with volume 1 on M. It is shown that the height is not bounded from below on this path if the dimension of M equals 0 or 1 modulo 4. Analogously, it is shown that for all s not covered by above existence results $\zeta(\cdot,s)$ is not bounded from below along this path. The final result regarding the existence of lower bounds is that the height and the ζ-function are always bounded from below if one restrics to the moduli space of Heisenberg-like metrics with volume 1 and sectional curvature bounded from above by a positive constant. In case M is a Heisenberg manifold, all of the above results concerning the existence of lower bounds are strengthened. Under the same assumptions the existence of global minima is concluded.

The last section treats concrete minima of the height and the ζ-function of Heisenberg manifolds. First, it is shown that that certain Heisenberg-type nilmanifolds in dimension 3, 5, 9 and 25 are (local) minima under variations of the metric in the space of Heisenberg-type metrics with volume 1. These nilmanifolds are characterised by the fact that the corresponding 2, 4, 8, resp. 24 dimensional base torus is the quotient by the hexagonal, checkerboard, $E8$-root resp. Leech lattice. The final result is that a global minimum of the height or the ζ-function of three dimensional Heisenberg manifolds is always attained at a metric for which the corresponding base torus is the quotient by a hexagonal lattice.

Zusammenfassung

Die vorliegende Arbeit untersucht die spektrale ζ-Funktion $\zeta((M,g),s)$ und die Höhe $\zeta'((M,g),0)$ kompakter nicht-singulärer heisenbergartiger Nilmannigfaltigkeiten (M,g). Heisenbergartige Nilmannigfaltigkeiten wurden von Gornet und Mast als natürliche Verallgemeinerung von Nilmannigfaltigkeiten vom Heisenberg-Typ eingeführt. Ist M eine Heisenbergmannigfaltigkeit $\Gamma\backslash H_n$, so ist jede von einer links-invarianten Metrik auf H_n induzierte Metrik heisenbergartig.

Das erste Resultat besteht aus Formeln für die spektrale ζ-Funktion und die Höhe einer heisenbergartigen Nilmannigfaltigkeit. Mithilfe dieser Formeln werden im Weiteren mehrere Ergebnisse zur (Nicht-)Existenz globaler unterer Schranken der Höhe $\zeta'(\cdot,0)$ und der ζ-Funktion $\zeta(\cdot,s)$ für positive s bewiesen. Das erste ist die Existenz unterer Schranken für Höhe und ζ-Funktion auf dem Modulraum der Metriken vom Heisenberg-Typ mit Volumen 1 einer gegebenen Nilmannigfaltigkeit M. Daraufhin wird die Existenz unterer Schranken der Höhe und der ζ-Funktion auf dem Modulraum der heisenbergartigen Metriken mit Volumen 1 über M untersucht. Solche existieren im Falle der Höhe, falls die Dimension von M gleich 3 modulo 4 beträgt. Im Falle von $\zeta(\cdot,s)$ hängt die Existenz von der Position von s ab.

Um die verbleibenden Dimensionen bzw. Positionen von s zu verstehen, wird ein Weg im Modulraum der heisenbergartigen Metriken mit Volumen 1 auf M eingeführt. Es wird gezeigt, dass, wenn die Dimension von M gleich 0 oder 1 modulo 4 ist, die Höhe auf diesem Weg nicht von unten beschränkt ist. Analog wird gezeigt, dass $\zeta(\cdot,s)$ für alle s, die von obigen Resultaten nicht abgedeckt wurden, auf diesem Weg nicht von unten beschränkt ist. Das finale Ergebnis zur Existenz globaler unterer Schranken ist dann, dass sowohl Höhe als auch ζ-Funktion stets von unten beschränkt sind, wenn man den Modulraum der heisenbergartigen Metriken mit Volumen 1 und von oben beschränkter Schnittkrümmung betrachtet.

Der letzte Abschnitt behandelt konkrete Minima der Höhe und ζ-Funktion von Heisenbergmannigfaltigkeiten. Zunächst wird gezeigt, dass in den Dimensionen 3, 5, 9 und 25 gewisse Heisenberg-Typ Mannigfaltigkeiten (lokale) Minima unter Variationen der Metrik im Raum der Heisenberg-Typ Metriken mit Volumen 1 sind. Diese zeichnen sich dadurch aus, dass der entsprechende zugehörige 2, 4, 8, bzw. 24 dimensionale Basistorus Quotient nach dem hexagonalen, Checkerboard-, E8-Wurzel- bzw. Leech-Gitter ist. Das finale Resultat ist, dass ein globales Minimum der Höhe bzw. der ζ-Funktion einer dreidimensionalen Heisenbergmannigfaltigkeit stets an einer Metrik angenommen wird, für die der zugehörige Basistorus der Quotient nach einem hexagonalen Gitter ist.

Contents

Introduction

Let M^n be a compact n-dimensional manifold and A a 2nd order elliptic self-adjoint operator on M. By elliptic PDE theory the spectrum of A is a pure point spectrum with only accumulation point ∞. Furthermore, every eigenvalue of A is real and of finite multiplicity. We assume further that A is bounded from below so that at most finitely many of its eigenvalues are negative. The spectral ζ-function of A is defined by

$$\zeta_A(s) := \sum_{\lambda \neq 0} \lambda^{-s}, \tag{0.1}$$

where the sum extends over all nonzero eigenvalues of A and the branch of λ^s is chosen so that $\lambda^s > 0$ when $\lambda > 0$ and $s \in \mathbb{R}$. The sum in (0.1) converges absolutely for $\Re s > n/2$ by known estimates for the growth of the eigenvalues of A. Furthermore, ζ_A extends to a meromorphic function on $\mathbb{C}$ which is regular at $s = 0$ (see **[See67]**). The value

$$\zeta_A'(0) = \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} \zeta_A(s) \tag{0.2}$$

is called the *height* of A. The associated value

$$\det A := e^{-\zeta_A'(0)} \tag{0.3}$$

is called the *determinant* of A. This naming can be justified as follows. The derivative of (0.1) for $\Re s$ large enough is given by

$$\frac{\mathrm{d}}{\mathrm{d}s} \zeta_A(s) = -\sum_{\lambda \neq 0} (\ln \lambda) \cdot \lambda^{-s}.$$

Formally setting $s = 0$ in this formula and substituting into (0.3) yields

$$e^{-\zeta_A'(0)} = \exp\left(\sum_{\lambda \neq 0} \ln \lambda\right) = \prod_{\lambda \neq 0} \lambda,$$

which can be seen as a generalisation of the well-known determinant formula from linear algebra. Of course, this calculation was purely formal. Indeed, the product on the right hand side does not converge. But we assign a finite value to it by the process of analytic continuation.

Usually, M comes equipped with a Riemannian metric g and $A = A_g$ is chosen to be a canonically given geometric operator such as the Laplace-Beltrami operator Δ_g, the

Hodge-Laplace operator $\Delta_{g,p}$ acting on exterior p-forms, the square of the Atiyah-Singer (Dirac) operator D_g or the conformal Laplacian L_g. In case of the Laplace-Beltrami operator we also speak of the height and the determinant of (M, g). Since $\det A_g$ is a spectral invariant, an important question is:

How is the geometry of (M, g) related to the determinant (of A_g)?

One line of work that explores this question examines the behaviour of the determinant under changes of the metric. More specifically, one relates the extrema of the determinant to the degree of symmetry of the metrics at which these are attained. For the determinant to possibly have any local extrema, one has to normalise the volume: a uniform scaling of the metric g by a constant τ results in A_g as above being scaled by τ^{-1}: $A_{\tau\cdot g} = \tau^{-1} \cdot A_g$. By formulas (0.2) and (0.3) we then have

$$\det A_{\tau g} = \tau^{-\zeta_{A_g}(0)} \cdot \det A_g \, .$$

It is very well known that in odd dimensions $\zeta_{A_g}(0) = -\dim\ker A_g$, whereas in even dimensions $\zeta_{A_g}(0)$ is related to the heat invariants of A_g. Thus, in general, a scaling of the metric results in a scaling of the determinant. As we will see shortly, there are cases where additional constraints have to be posed in order for the determinant to have *global* extrema.

In **[OPS88b]**, B. Osgood, R. Phillips and P. Sarnak investigated the extremal values of the determinant of the Laplace-Beltrami operator Δ_g on compact smooth surfaces (M, g). If $\partial M \neq \emptyset$ we impose Dirichlet boundary conditions.

In case M is closed we call g *uniform* if it has constant curvature. In case $\partial M \neq \emptyset$ the metric g is called *uniform* if one of the following conditions holds:

(I) (M, g) has constant curvature and ∂M has zero geodesic curvature.

(II) (M, g) is flat and ∂M is of constant geodesic curvature.

Metrics satisfying (I) or (II) are called metrics of type I and II, respectively. We further introduce the following conditions (CI) and (CII) for metrics g in case $\partial M \neq \emptyset$:

(CI) $\int\limits_{\partial M} k \mathrm{d}s \geqslant 0$, or equivalently $\int\limits_{M} K \mathrm{d}A \leqslant 2\pi\chi(M)$,

(CII) $\int\limits_{\partial M} k \mathrm{d}s \geqslant 2\pi\chi(M)$ or $\int\limits_{M} K \mathrm{d}A \leqslant 0$.

Here, k, K, $\mathrm{d}s$ and $\mathrm{d}A$ denote the geodesic curvature, the Gauss curvature, the arc length and the area, respectively, relative to the metric g, and $\chi(M)$ is the Euler characteristic of the surface.

THEOREM 0.1 ([**OPS88b**][Theorem 1]).

(a) If M is closed then of all metrics in a given conformal class and of given area, the uniform metric has maximum determinant.

(b) If $\partial M \neq \emptyset$ then in a given conformal class of metrics all of a given area and all satisfying (CI), the uniform metric of type I has maximum determinant.

(c) If $\partial M \neq \emptyset$ then in a given conformal class of metrics all of a given boundary length and all satisfying (CII), the uniform metric of type II has maximum determinant.

Note that in case M is closed the assertion that a uniform metric exists in a given conformal class is equivalent to the uniformisation theorem.

On the sphere S^2 every metric is conformally equivalent to the round metric which by Theorem 0.1 maximises the determinant. This particular case was already solved in [**Ono82**].

On the two-torus T^2 there is a manifold of flat metrics and the metric which maximises the determinant in the class of flat metrics with a given area maximises the determinant among all metrics with a given area. Regarding the determinant of flat metrics on T^2, B. Osgood, R. Phillips and P. Sarnak obtained the following result.

COROLLARY 0.2 ([**OPS88b**][Corollary 1(b)]). *In the class of flat metrics of volume 1 on the two-torus T^2 the determinant is maximised by the flat metric corresponding to the hexagonal lattice.*

We note that this result is implicit in earlier work on Epstein's ζ-function (see Theorem 2.43 and Remark 2.44*(ii)*).

We stay in the realm of flat tori as it relates directly to the results in this thesis. The results are formulated in terms of the *height*. By (0.3) its minima are precisely the maxima of the determinant.

P. Chiu proved that the height attains a global minimum on the moduli space of flat n-dimensional tori with volume 1 [**Chi97**, Theorem 3.1]. The minima of the height of flat tori $T = L\backslash\mathbb{R}^n$, where $L \subset \mathbb{R}^n$ is a lattice of full rank, are related to the *minimal norm* $m(L)$ of L, which is the minimal squared length of a nonzero vector in L:

$$m(L) = \min\{\|v\|^2 \mid v \in L\backslash\{0\}\}\,.$$

Note that this is exactly the squared systole $\mathrm{sys}(T)^2$ of T, i.e., the squared length of a shortest noncontractible closed curve in T. Maximising $m(L)$ in the space of volume normalised lattices is the much studied and well-known *lattice packing problem*. The solutions L_n are known and unique in dimensions $n = 2, \ldots, 8$ and $n = 24$ (see Definition and Remarks 2.39) and have particularly large automorphism groups. For example,

L_2 is the familiar hexagonal lattice whose automorphism group is a dihedral group of order 12 generated by a rotation through 60° and a reflection in a line through 0 joining two points of L_2 of minimal distance. Note that the corresponding torus $L_2\backslash\mathbb{R}^2$ is the global minimiser of the height (Corollary 0.2).

P. Sarnak and A. Strömbergsson [**SS06**] looked at the height of higher dimensional flat tori. They showed that $L_3\backslash\mathbb{R}^3$ is the unique global minimiser of the height in dimension three, thereby improving a local result of V. Ennola [**Enn64b**] and a result of P. Chiu [**Chi97**], who checked the height of $L_3\backslash\mathbb{R}^3$ against the height of a set of lattices equidistributed in the moduli space of three dimensional flat tori of volume 1. P. Sarnak and A. Strömberggson also proved that in dimensions $n = 4, 8, 24$, the height has a strict local minimum at $L_n\backslash\mathbb{R}^n$ with L_n as above.

As for the general case, Sarnak conjectured that the height on the moduli space of flat tori with volume 1 is always globally minimised at a torus $L_n\backslash\mathbb{R}^n$, where L_n is a lattice with a longest shortest vector, i.e., a lattice which maximises $m(L)$ (see Conjecture 2.47).

The work of B. Osgood, R. Phillips and P. Sarnak on the height/determinant stimulated a lot of research. They themselves followed up with two articles ([**OPS89**], [**OPS88a**]) proving that on any given compact surface, the set of metrics isospectral to a given metric is compact in the C^∞ topology. Here, on a surface with boundary, Dirichlet boundary conditions are imposed and the result is directly related to M. Kac' question "Can one hear the shape of a drum?". An essential ingredient in their proof was the use of the determinant.

K. Richardson investigated the determinant with respect to conformal variations in higher dimensions [**Ric94**]. He obtained a necessary criterion for a metric to be a critical point of the determinant under such variations, which is satisfied by homogeneous metrics. Complementing this result, Richardson gives examples of locally homogeneous spaces which are not critical. In dimension three, he also computes the second derivative of the determinant and gives examples of local maxima under conformal variations, among which are the flat torus $\mathbb{Z}^3\backslash\mathbb{R}^3$ and the round sphere S^3.

In dimension 4 and with respect to conformal variations of the metric, the determinant of the conformal Laplacian L and the square D^2 of the Dirac operator D was studied in [**BØ91**], [**BCY92**], [**CY95**] and [**Gur97**]. For example, the round sphere S^4 is a maximum in its conformal class for $\det D^2$ and a minimum for $\det L$ (see [**BCY92**]). In [**Bra93**] T. P. Branson proved that the round sphere S^6 is a maximum for the $\det L$ and a minimum for $\det D^2$ in its conformal class, which led him to conjecture that the pattern continues, i.e., that the round sphere S^n is a local minimum for $(-1)^{\frac{n}{2}} \det L$ and a local maximum for $(-1)^{\frac{n}{2}} \det D^2$ in its conformal class for all even $n \geqslant 2$.

A truly novel approach was made by K. Okikiolu **[Oki01]**, who worked out formulas for the first and second variation of $\det \Delta$ and $\det L$ under volume preserving variations in the space of all Riemannian metrics over a compact odd dimensional manifold. With these formulas, she showed that S^3 is a local maximum for $\det \Delta$, generalising K. Richardson's result across conformal classes. She also showed that for $m \geqslant 1$, S^{4m+3} is a saddle point for $\det \Delta$ among metrics fixing the conformal class. In contrast, S^{4m+1} is a local minimum and S^{4m+3} a local maximum for $\det L$ in the space of all Riemannian metrics. Note that this, in particular, shows the analogue of T. P. Branson's conjecture in odd dimensions, though the statement is of course much stronger. N. M. Møller in **[Mø12]** and B. Ørsted and N. M. Møller in **[MØ14]** completed this line of work by showing that for all $n \geqslant 2$, S^n is a local minimum for $(-1)^{\lfloor \frac{n}{2} \rfloor} \det L$ and a local maximum for $(-1)^{\lfloor \frac{n}{2} \rfloor} \det D^2$ in the space of all Riemannian metrics with the same volume. In particular, they confirmed T. P. Branson's conjecture.

Thereby, we finish our (nonexhaustive) account of the literature surrounding extremal properties of determinants and turn to the present work.

This thesis contributes to the theory of the behaviour of the spectral ζ-function and the height of the Laplace-Beltrami operator under metric variations in a class of locally homogeneous manifolds. More specifically, we deal with compact nilmanifolds $(\Gamma\backslash G, \mathbf{m})$. Here, G is a simply connected 2-step nilpotent Lie group, $\Gamma \subset G$ a discrete and cocompact subgroup and $\mathbf{m}$ is induced by a left invariant metric on G. The prototypical example is $G = H_n$, the $(2n+1)$-dimensional Heisenberg group. For other groups, we impose additional regularity assumptions on the metric $\mathbf{m}$, namely that it is Heisenberg-like as defined by R. Gornet and M. Mast in **[GM00]**. An important subclass of the manifolds we consider is that of Heisenberg-type nilmanifolds. The compact 2-step nilmanifolds are precisely the total spaces of principal torus-bundles over tori. Thus, it is clear from the beginning that there is some relation to the ζ-function and the height of flat tori.

Our first result establishes formulas for the ζ-function $\zeta((\Gamma\backslash G, \mathbf{m}), s)$ (Theorem 2.5) and the height $\zeta'((\Gamma\backslash G, \mathbf{m}), 0)$ (Corollary 2.6). Then, we investigate conditions under which $\mathbf{m} \mapsto \zeta((\Gamma\backslash G, \mathbf{m}), s)$ with $s > 0$ not a pole and $\mathbf{m} \mapsto \zeta'((\Gamma\backslash G, \mathbf{m}), 0)$ are bounded from below or attain global minima. Theorem 2.10 states that both functions are always bounded from below when we restrict to the space of volume normalised metrics of Heisenberg-type. In case the centre of G is 1-dimensional, they even attain global minima. Next, we show that if $\dim(\Gamma\backslash G) \equiv 3 \pmod 4$, then the height is bounded from below on the moduli space of volume normalised Heisenberg-like metrics (Theorem 2.11). The existence of lower bounds for the ζ-function depends on the position of s. In case the centre of G is 1-dimensional, both results are strengthened as above: under the same assumptions, the height and the ζ-function attain a global minimum. To understand whether the respective restrictions on the dimension and the position of s are necessary, we introduce a path of metrics in the space of volume normalised Heisenberg-like metrics (Proposition 2.12). If $\dim(\Gamma\backslash G) \equiv 0, 1 \pmod 4$ then the height is not bounded from below on this path (Theorem 2.19). Also, for all positions s that were not covered by Theorem 2.11, the ζ-function is not bounded from below along this path. In case the group G is the $(2n+1)$-dimensional Heisenberg group we introduce a second path in the space of volume normalised metrics. This path is geometrically more restrictive than the first one. Nevertheless, if $\dim H_n \equiv 1 \pmod 4$ then the height is not bounded from below along it (Theorem 2.28). A similar result holds for the ζ-function.

The two paths of metrics share the property that the sectional curvature along them is not bounded from above. As it turns out, this is precisely the reason for the nonexistence of lower bounds for the height and the ζ-function in the situations of Theorem 2.19 and Theorem 2.28: the height and ζ-function are bounded from below on the moduli space of volume normalised Heisenberg-like metrics whose sectional curvature is bounded from above by a positive constant (Theorem 2.30). As above, both functions attain a global minimum if the centre of G is 1-dimensional.

Unfortunately, the existence of lower bounds for the height in case $\dim(\Gamma\backslash G) \equiv 2 \pmod 4$ eludes our analysis. Nevertheless, the existence of lower bounds obeys a $\pmod 4$-pattern w.r.t. the dimension. As we have seen above, such patters are also present in work of K. Okikiolu, N. M. Møller and B. Ørstedt ([**Oki01**], [**Mø12**], [**MØ14**]). However, the author does not know how they are related to our $\pmod 4$-pattern since the above mentioned authors deal with different settings and methods.

We also investigate concrete minima of the height and the ζ-function in the class of Heisenberg manifolds $(\Gamma\backslash H_n, \mathbf{m})$. In dimension 3 this is no restriction: every compact 2-step nilmanifold of dimension 3 is a Heisenberg manifold. A minimum of the height or the ζ-function in the class of 3-dimensional Heisenberg-type nilmanifolds is always attained at a metric at which the 2-dimensional base torus, corresponding to the principal torus-bundle structure, is the quotient by a hexagonal lattice (Theorem 2.49). Note that the existence of these minima is assured by Theorem 2.10. We also show that for $n \in \{2, 4, 12\}$ certain $(2n+1)$-dimensional Heisenberg-type nilmanifolds are local minima in the class of Heisenberg-type nilmanifolds. Analogous to the 3-dimensional case, their base torus is of the form $L_{2n}\backslash\mathbb{R}^{2n}$, where L_{2n} is a solution of the lattice packing problem.

Coming back to general Heisenberg manifolds, we prove that for a metric $\mathbf{m}$ to be extremal for the height or the ζ-function of $(\Gamma\backslash H_n, \mathbf{m})$, the height of the corresponding base torus must be extremal on a certain submanifold of the moduli space of volume normalised flat tori (Theorem 2.50). This allows us to prove that every global minimum of the height or the ζ-function in dimension 3 is attained at a metric for which the corresponding base torus is the quotient by a hexagonal lattice (Theorem 2.52). We finish with computations that suggest that a certain 5-dimensional Heisenberg manifold is extremal for the height (pp. 117).

This thesis is organised as follows. In Chapter 1, we lay the ground for our work on the height. Section 1 introduces the reader to (2-step) nilmanifolds, their spectra and geometry. In the following three sections, we state basic facts about the manifold of positive definite symmetric matrices, Schwartz functions and their Fourier transform, and parameter dependent integrals. These sections serve as reference and to fix notation and conventions. In Section 5 we introduce the various moduli spaces that we will use. The main result of this section, which might also be of independent interest, is a characterisation of the compact sets of the moduli space of all metrics of a Heisenberg manifold (Theorem 1.63 and Corollary 1.64). Section 6 focuses on the heat trace of Heisenberg-like nilmanifolds. We prove a Poisson type formula (Theorem 1.78 and Proposition 1.82) which provides, in particular, the explicit description of the asymptotic expansion of the heat trace (Corollary 1.83 and Lemma 1.90).

Chapter 2 contains the main results of this thesis. In Section 1 we prove Theorem 2.5 and Corollary 2.6 stating our formulas for the ζ-function and the height. Section 2 is devoted to the study of the global behaviour of the height and the ζ-function. Here, we prove Theorems 2.10, 2.11, 2.19, 2.28 and Theorem 2.30. We finish this section by considering discrete sequences of $(2n+1)$-dimensional Heisenberg-type manifolds whose height was investigated by K. Furutani and S. de Gosson **[Fd03]** in the cases $n = 1, 2$, see Example 2.31 and Proposition 2.32. Section 3 looks at concrete minima of the height and the ζ-function. In Subsection 3.1, we introduce Epstein's ζ-function, which is closely related to the spectral ζ-function of flat tori. We also describe the lattice packing problem and relate its solutions to the minimisers of Epstein's ζ-function and the height of flat tori. In the last section, Subsection 3.2, we prove Theorems 2.49, 2.50 and Theorem 2.52. We close with said computations for the height of a 5-dimensional Heisenberg manifold.

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CHAPTER 1

Preliminaries

1. Structure of compact 2-step Nilmanifolds

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. The *descending central series* of $\mathfrak{g}$ is defined inductively by

$$\begin{aligned}\mathfrak{g}^{(1)} &:= \mathfrak{g}\\ \mathfrak{g}^{(n+1)} &:= [\mathfrak{g}, \mathfrak{g}^{(n)}] := \operatorname{span}_{\mathbb{R}} \left\{ [X, Y] \mid X \in \mathfrak{g}, Y \in \mathfrak{g}^{(n)} \right\} .\end{aligned}$$

With this definition we have for all $p, q \in \mathbb{N}$:

$$\begin{aligned}[\mathfrak{g}^{(p+1)}, \mathfrak{g}^{(q)}] = [[\mathfrak{g}, \mathfrak{g}^{(p)}], \mathfrak{g}^{(q)}] &\subseteq [\mathfrak{g}^{(p)}, [\mathfrak{g}, \mathfrak{g}^{(q)}]] + [\mathfrak{g}, [\mathfrak{g}^{(p)}, \mathfrak{g}^{(q)}]]\\ &\subseteq [\mathfrak{g}^{(p)}, \mathfrak{g}^{(q+1)}] + [\mathfrak{g}, \mathfrak{g}^{(p+q)}] = \mathfrak{g}^{(p+q+1)}\end{aligned}$$

by Jacobi's identity and induction. In particular, each $\mathfrak{g}^{(n)}$ is an ideal.

The Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if there is $n \in \mathbb{N}$ such that $\mathfrak{g}^{(n+1)} = \{0\}$. If n is the smallest such integer then $\mathfrak{g}$ is said to be *n-step nilpotent*. Note that if $\mathfrak{g}$ is n-step nilpotent then $\mathfrak{g}^{(n)} \subseteq \mathfrak{z}$, where $\mathfrak{z} := \mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0\, \forall Y \in \mathfrak{g}\}$ is the centre of $\mathfrak{g}$. In particular, a nontrivial nilpotent Lie algebra always has nontrivial centre. Let now G be a Lie group with Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is (n-step) nilpotent, we call G (n-step) nilpotent as well. *From now on, unless stated otherwise, we will always assume that G is connected, simply connected and 2-step nilpotent*. This means that for all $X, Y \in \mathfrak{g}$ we have $[X, Y] \in \mathfrak{z}$. The Lie algebra $\mathfrak{g}$ is, so to speak, as close to being abelian as possible.

For any Lie group H with Lie algebra $\mathfrak{h}$ there is an open neighborhood $U \subseteq \mathfrak{h}$ of 0 such that the Lie group exponential map $\exp : \mathfrak{h} \to H$ restricted to U is a diffeomorphism onto its image. We denote by $\log : \exp(U) \to U$ the inverse and define

$$X * Y := \log(\exp X \cdot \exp Y)$$

for all $X, Y \in U$ such that $\exp X \cdot \exp Y \in \exp U$. The Campbell-Baker-Hausdorff formula expresses the product $X * Y$ in terms of algebraic operations in $\mathfrak{h}$. In case H is 2-step nilpotent, it reads

$$X * Y = X + Y + \frac{1}{2}[X, Y] . \tag{1.1}$$

Restricting to connected and simply-connected 2-step nilpotent Lie groups gives us the following structure theorem.

THEOREM 1.1 ([**CG90**, Theorem 1.2.1]). *Let G be a connected and simply connected 2-step nilpotent Lie group with Lie algebra $\mathfrak{g}$.*

(a) $\exp : \mathfrak{g} \to G$ *is an analytic diffeomorphism.*

(b) The Campbell-Baker-Hausdorff formula (1.1) *holds for all $X, Y \in \mathfrak{g}$.*

We can thus think of G as $\mathfrak{g}$ with multiplication $*$ as above. Another consequence is that the centre $\mathfrak{Z}$ of G is precisely $\mathfrak{Z} = \exp(\mathfrak{z})$. The following example is paramount for everything to come.

EXAMPLE 1.2. Let $n \in \mathbb{N}$. We define $\mathfrak{h}_n$, the $(2n+1)$-dimensional *Heisenberg algebra*, to be the Lie algebra with basis $\mathfrak{B}_n := (X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z)$. The Lie bracket is given by the following relations on the basis vectors:

$$\begin{aligned} [X_i, Y_j] &= -[Y_j, X_i] = \delta_{i,j} Z && 1 \leqslant i, j \leqslant n\,, \\ [X_i, X_j] &= [Y_i, Y_j] = 0 && 1 \leqslant i, j \leqslant n\,, \\ [Z, X_i] &= [Z, Y_i] = 0 && 1 \leqslant i \leqslant n\,. \end{aligned}$$

Clearly, the centre $\mathfrak{z}$ of $\mathfrak{h}_n$ is spanned by Z. We realise $\mathfrak{h}_n$ as a matrix algebra. With $\sum_{j=0}^{n} \left(x_j X_j + y_j Y_j \right) + sZ$, where $x, y \in \mathbb{R}^n$ and $s \in \mathbb{R}$, we associate the matrix

$$X(x, y, s) := \begin{pmatrix} 0 & x^t & s \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \in M(n+2; \mathbb{R}). \tag{1.2}$$

The Lie bracket in this representation is given by the comutator, i.e.,

$$\begin{aligned} [X(x, y, s), X(x', y', s')] &= X(x, y, s) X(x', y', s') - X(x', y', s') X(x, y, s) \\ &= X(0, 0, A((x, y), (x', y')))\,, \end{aligned}$$

where A is the standard symplectic form on $\mathbb{R}^{2n}$ whose matrix representation in the standard basis is

$$J = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}. \tag{1.3}$$

Note that $\mathfrak{B}_n$ is the standard basis of $\mathbb{R}^{2n+1}$ in the representation (1.2).

The group G corresponding to the Heisenberg algebra $\mathfrak{g} = \mathfrak{h}_n$ is the $(2n+1)$ dimensional Heisenberg group H_n. For $x, y \in \mathbb{R}^n$ and $s \in \mathbb{R}$ let

$$\gamma(x,y,s) := \begin{pmatrix} 1 & x^t & s \\ 0 & \mathrm{Id}_n & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(n+2;\mathbb{R})\,. \tag{1.4}$$

Then H_n is defined by

$$H_n := \{\gamma(x,y,s) \mid x, y \in \mathbb{R}^n, s \in \mathbb{R}\} \tag{1.5}$$

with the group structure it inherits from $\mathrm{GL}(n+2;\mathbb{R})$. The Lie group exponential map is the matrix exponential map:

$$\begin{aligned} \exp X(x,y,s) &= Id + X(x,y,s) + \frac{1}{2}X(x,y,s)^2 \\ &= \gamma\left(x,y,s+\tfrac{1}{2}\langle x,y\rangle\right) \end{aligned} \tag{1.6}$$

and

$$\log\gamma(x,y,s) = X\left(x,y,s-\tfrac{1}{2}\langle x,y\rangle\right)\,, \tag{1.7}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard euclidean inner product.

We note that every nilpotent Lie algebra can be realised as a matrix algebra of upper triangular matrices. This is known as the Birkhoff Embedding Theorem, see [**CG90**, Theorem 1.1.11].

We now equip $\mathfrak{g}$ with a positive definite inner product $\mathbf{m} = \langle\cdot,\cdot\rangle_{\mathbf{m}}$ and extend it to a left invariant Riemannian metric on G, which we also denote by $\mathbf{m}$, by the usual identification $T_eG = \mathfrak{g}$. We denote the norm corresponding to the inner product $\langle\cdot,\cdot\rangle_{\mathbf{m}}$ by $\|\cdot\|_{\mathbf{m}}$. Let $\mathfrak{n} := \mathfrak{n}(\mathfrak{g}) := \mathfrak{z}^\perp$ be the orthogonal complement of $\mathfrak{z}$ w.r.t. $\mathbf{m}$, i.e.,

$$\mathfrak{g} = \mathfrak{z} \ominus \mathfrak{n}\,, \tag{1.8}$$

and write $\mathbf{m}_{\mathfrak{z}}$ and $\mathbf{m}_{\mathfrak{n}}$ for the restriction of $\mathbf{m}$ to $\mathfrak{z}$ and $\mathfrak{n}$ respectively. Define the *structure map* $j = j_{\mathbf{m}} : \mathfrak{z} \to \mathfrak{so}(\mathfrak{n})$ by

$$\langle j(Z)X, Y\rangle_{\mathbf{m}} = \langle Z, [X,Y]\rangle_{\mathbf{m}} \quad \forall Z \in \mathfrak{z},\ X, Y \in \mathfrak{n}\,. \tag{1.9}$$

The map j is well-defined by (1.9). Conversely, given two euclidean vector spaces $(\mathfrak{z}, \mathbf{m}_{\mathfrak{z}})$ and $(\mathfrak{n}, \mathbf{m}_{\mathfrak{n}})$ and a nontrivial linear map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{n})$, we can define a metric 2-step nilpotent Lie algebra $(\mathfrak{g}, \mathbf{m})$ as follows: Define $\mathfrak{g}$ as a euclidean vector space by (1.8) and an alternating bracket $[\cdot,\cdot] : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{z}$ by (1.9). Extend $[\cdot,\cdot]$ to all of $\mathfrak{g} \times \mathfrak{g}$ by $[Z, X] = 0$ for all $Z \in \mathfrak{z}, X \in \mathfrak{g}$. Note that $\mathfrak{z}$ is not neccessarily the centre of $\mathfrak{g}$ in this construction. It is, however, if the map $j(Z)$ has trivial kernel for all $Z \in \mathfrak{z}\backslash\{0\}$.

By Theorem 1.1 and the fact that the Riemannian metric $\mathbf{m}$ of G is left-invariant, it is apparent that the geometry of $(G, \mathbf{m})$ is completely encoded in j.

We introduce important classes of 2-step nilpotent Lie algebras which possess certain degrees of regularity. Note that $j(Z)$ is skew-symmetric w.r.t. $\mathbf{m}$ for each $Z \in \mathfrak{z}$ since $[\cdot, \cdot]$ is alternating. It follows that the eigenvalues of $j(Z)$ are purely imaginary and come in complex conjugate pairs, except for a possible eigenvalue 0.

DEFINITION 1.3.

(i) The metric Lie algebra $(\mathfrak{g}, \mathbf{m})$ is called *nonsingular* if $j(Z)$ has trivial kernel for all $Z \in \mathfrak{z}\backslash\{0\}$. Note that in this case $\dim \mathfrak{n} = 2n$ for some $n \in \mathbb{N}$. We call $(\mathfrak{g}, \mathbf{m})$ *singular* if it is not nonsingular.

(ii) We call a nonsingular metric 2-step nilpotent Lie algebra $(\mathfrak{g}, \mathbf{m})$ a *Heisenberg-like Lie algebra* if there are real numbers $0 < c_1^{\mathbf{m}} \leqslant \ldots \leqslant c_n^{\mathbf{m}}$ such that for all $Z \in \mathfrak{z}$ the eigenvalues of $j(Z)$ are $\pm i c_j^{\mathbf{m}} \|Z\|_{\mathbf{m}}$ with $1 \leqslant j \leqslant n$.

(iii) A Heisenberg-like Lie algebra $(\mathfrak{g}, \mathbf{m})$ is a *Heisenberg type Lie algebra* or *H-type algebra* if $c_1^{\mathbf{m}} = c_2^{\mathbf{m}} = \ldots = c_n^{\mathbf{m}} = 1$, i.e., if for each $Z \in \mathfrak{z}$ the eigenvalues of $j(Z)$ are $\pm i\|Z\|_{\mathbf{m}}$.

(iv) We call $(G, \mathbf{m})$ a nonsingular, Heisenberg-like or Heisenberg type nilpotent Lie group if $(\mathfrak{g}, \mathbf{m})$ has the corresponding property. Also, if the group or the Lie algebra is fixed we call $\mathbf{m}$ a Heisenberg-like or Heisenberg type metric.

REMARK 1.4.

(i) The metric Lie algebra $(\mathfrak{g}, \mathbf{m})$ being nonsingular is equivalent to $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{z}$, $\mathrm{ad}_X(\cdot) = [X, \cdot]$, being surjective for all $X \in \mathfrak{g}\backslash\mathfrak{z}$. This criterion does not involve the metric $\mathbf{m}$. In fact, if Definition 1.3.(i) is fulfilled for one inner product on $\mathfrak{g}$, it is so for any inner product on $\mathfrak{g}$ (see Definition 1.4 and Lemma 1.8 in [**Ebe94**]). Consequently, we call a 2-step nilpotent Lie group G with Lie algebra $\mathfrak{g}$ nonsingular, if $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{z}$ is surjective for all $X \in \mathfrak{g}\backslash\mathfrak{z}$.

(ii) Every 2-step nilpotent Lie algebra $\mathfrak{g}$ with $\dim \mathfrak{z} = 1$ is isomorphic to the Heisenberg algebra $\mathfrak{h}_n$, $n = (\dim \mathfrak{g} - 1)/2$.

(iii) Let $\mathbf{m}$ be an inner product on the Heisenberg algebra $\mathfrak{h}_n$. Since $\mathfrak{z}$ is one dimensional, there exist numbers $0 < c_1^{\mathbf{m}} \leqslant c_2^{\mathbf{m}} \leqslant \ldots c_n^{\mathbf{m}}$ such that for every $Z \in \mathfrak{z}$ the eigenvalues of $j(Z)$ are $\pm i c_j^{\mathbf{m}} \|Z\|_{\mathbf{m}}$. In particular, $(\mathfrak{h}_n, \mathbf{m})$ is a nonsingular Heisenberg-like Lie algebra.

(iv) The Heisenberg-like Lie algebras were introduced in [**GM00**]. However, there a slightly broader definition was made which included certain singular Lie algebras. We will at no point deal with singular Lie algebras.

REMARK 1.5. The Heisenberg type Lie algebras were introduced in [**Kap81**]. Let $(\mathfrak{g}, \mathbf{m})$ be such a Lie algebra. By definition we have

$$j(Z)^2 = -\|Z\|^2_{\mathbf{m}} \cdot \mathrm{Id} \quad \text{for all } Z \in \mathfrak{z}\,. \tag{1.10}$$

Now recall the definition of the (real) Clifford algebra $Cl(\mathfrak{z}, \|\cdot\|^2_{\mathbf{m}_\mathfrak{z}})$ over the quadratic space $(\mathfrak{z}, \|\cdot\|^2_{\mathbf{m}_\mathfrak{z}})$: it is the (up to isomorphism) unique associative unital algebra containing $\mathfrak{z}$ subject to the relations

$$X \cdot X = -\|X\|^2_{\mathbf{m}_\mathfrak{z}} \cdot 1 \quad \text{for all } X \in \mathfrak{z}\,.$$

Clearly $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{n})$ extends to a unitary (algebra) representation of $Cl(\mathfrak{z}, \|\cdot\|^2_{\mathbf{m}_\mathfrak{z}})$. By unitary, we mean that every $Z \in \mathfrak{z} \subset Cl(\mathfrak{z}, \|\cdot\|^2_{\mathbf{m}_\mathfrak{z}})$ with $\|Z\|_{\mathbf{m}_\mathfrak{z}} = 1$ acts orthogonally. Conversely, if $(\mathfrak{n}, \|\cdot\|^2_{\mathbf{m}_\mathfrak{n}})$ is the representation space of a unitary $Cl(\mathfrak{z}, \|\cdot\|^2_{\mathbf{m}_\mathfrak{z}})$-representation, the inclusion map $\mathfrak{z} \hookrightarrow Cl(\mathfrak{z}, \|\cdot\|^2_{\mathbf{m}_\mathfrak{z}})$ induces a map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{n})$ which satisfies (1.10).

The Clifford algebras and their representations are classified (see, e.g., [**LM89**, Chapter I, §§ 3-5]). Each representation is a direct sum of irreducible representations. The number ν_l of irreducible (real) representations of the (real) Clifford algebra $Cl(\mathfrak{z}, \|\cdot\|^2_{\mathbf{m}_\mathfrak{z}})$ with $\dim \mathfrak{z} = l$ is $\nu_l = 1$ if $l \not\equiv 3 \pmod 4$ and $\nu = 2$ if $l \equiv 3 \pmod 4$.

EXAMPLE 1.6. Let $\mathfrak{z} := \mathbb{R}^2$ be equipped with the standard euclidean inner product. The Clifford algebra $Cl(\mathfrak{z}, \|\cdot\|^2_{std})$ in this case is simply denoted by Cl_2. As algebras we have $Cl_2 \cong \mathbb{H}$, the quaternions (see [**LM89**, Chapter I, § 4]). Up to isomorphism, there is only one irreducible real representation of Cl_2. This is apparent from the fact that $\mathbb{H}$ is a simple algebra and its unique irreducible representation can be realised by left multiplication on itself. Any Cl_2-module $(\mathfrak{n}, \|\cdot\|^2_{\mathbf{m}_\mathfrak{n}})$ must be a direct sum of copies of this irreducible module. In particular, a metric 2-step nilpotent Heisenberg-type Lie algebra $(\mathfrak{g}, \mathbf{m})$ with $\dim \mathfrak{z} = 2$ has $\dim \mathfrak{n} = 4k$, $k \in \mathbb{N}$.

Concretely, let $(\mathfrak{n}, \|\cdot\|^2) := (\mathbb{R}^4, \|\cdot\|^2_{std})$ and define $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{n})$ in the standard coordinate system by

$$j(e_1) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad j(e_2) := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{1.11}$$

and linear continuation. Then the eigenvalues of $j(Z)$ are $\pm i\|Z\|$ for all $Z \in \mathfrak{z}$. Hence, the corresponding 2-step nilpotent Lie algebra is of Heisenberg type. In particular, for any metric Heisenberg type Lie algebra $(\mathfrak{g}, \mathbf{m})$ with $\dim \mathfrak{z} = 2$ and structure map $j' : \mathfrak{z} \to \mathfrak{so}(\mathfrak{n})$, $j'(Z)$ is (up to a metric Lie algebra isomorphism) equivalent to the direct sum of $j(Z) \oplus \ldots \oplus j(Z)$ for all $Z \in \mathfrak{z}$, with j as in (1.11).

EXAMPLE 1.7. A slight alteration of the previous example yields a Heisenberg-like Lie algebra. Let $(\mathfrak{z}, \mathbf{m}_{\mathfrak{z}}) = (\mathbb{R}^2, \langle\cdot,\cdot\rangle_{std})$, $(\mathfrak{n}, \mathbf{m}_{\mathfrak{n}}) = (\mathbb{R}^4, \langle\cdot,\cdot\rangle_{std})$ and $\alpha > 0$. Define $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{n})$ in standard coordinates by

$$j(e_1) := \begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\alpha \\ 0 & 0 & -1/\alpha & 0 \end{pmatrix}, \quad j(e_2) := \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -1/\alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & 1/\alpha & 0 & 0 \end{pmatrix}$$

and linear continuation. Then the eigenvalues of $j(Z)$ are $\pm i\alpha\|Z\|, \pm i\frac{1}{\alpha}\|Z\|$ for all $Z \in \mathfrak{z}$. Hence, the corresponding metric 2-step nilpotent Lie algebra is Heisenberg-like. Note that there is no structure theory for Heisenberg-like Lie algebras as there is for Heisenberg type Lie algebras.

It is time to come to our chosen objects of study. Let $\Gamma \subset G$ be a discrete and cocompact subgroup of G. We call such Γ *uniform*. They exist if and only if the Lie algebra $\mathfrak{g}$ has a basis $(X_1, \ldots, X_{\dim \mathfrak{g}})$ such that the corresponding structure constants $a_{i,j}^k$, defined by $[X_i, X_j] = \sum a_{i,j}^k X_k$, are rational (see, e.g., **[CG90**, Theorem 5.1.8]). For example, this is always the case for Heisenberg type Lie algebras (see **[CD02]**). The quotient $\Gamma\backslash G$ is, as a smooth manifold, the total space of a principal torus-bundle over a torus. Conversely, every such space is the compact quotient of a 2-step nilpotent Lie group **[PS61]**.

PROPOSITION 1.8. *Let G be a connected and simply-connected nonsingular 2-step nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $\Gamma \subset G$ a uniform subgroup. Let $\ell := \dim \mathfrak{z}$ and $n := (\dim \mathfrak{g} - \ell)/2$. Then there exists a basis $(Z_1, \ldots, Z_\ell, X_1, \ldots, X_{2n})$ of $\mathfrak{g}$ such that $(Z_1, \ldots, Z_\ell)$ is a basis of $\mathfrak{z}$ and*

$$\Gamma = \left\{ \exp(m_1 Z_1) \cdots \exp(m_\ell Z_\ell) \cdot \exp(m_{\ell+1} X_1) \cdots \exp(m_{\ell+2n} X_{2n}) \mid m \in \mathbb{Z}^{\ell+2n} \right\}.$$

PROOF. This is a special case of **[CG90**, Proposition 5.3.2] with $H_1 := \mathfrak{Z}$, the centre of G, and $\mathfrak{h}_1 = \mathfrak{z}$, the Lie algebra of $\mathfrak{Z}$. □

Since Γ acts on G by left translation, its action is isometric w.r.t. the metric $\mathbf{m}$. There is thus a unique Riemannian metric on $\Gamma\backslash G$, which we also denote by $\mathbf{m}$, such that the canonical projection $\pi : (G, \mathbf{m}) \to (\Gamma\backslash G, \mathbf{m})$ is a Riemannian covering. We call $(\Gamma\backslash G, \mathbf{m})$ a *compact Riemannian (2-step) nilmanifold*. In case $(G, \mathbf{m})$ is a nonsingular, Heisenberg-like or a Heisenberg type nilpotent Lie group, we say that $(\Gamma\backslash G, \mathbf{m})$ is a nonsingular, Heisenberg-like or Heisenberg type nilmanifold, respectively. Also, if $G = H_n$, the $(2n+1)$-dimensional Heisenberg group, we call $(\Gamma\backslash H_n, \mathbf{m})$ a *compact Heisenberg manifold*. By Remark 1.4*(iii)* a compact Heisenberg manifold is always a nonsingular Heisenberg-like nilmanifold.

Denote by $\pi_{\mathfrak{n}} : \mathfrak{g} \to \mathfrak{n}$ the orthogonal projection onto $\mathfrak{n}$. Define the two sets

$$\begin{aligned} \Gamma_{\mathfrak{z}} &:= \log \Gamma \cap \mathfrak{z} \,, \\ \Gamma_{\mathfrak{n}} &:= \pi_{\mathfrak{n}} \log \Gamma \,. \end{aligned} \tag{1.12}$$

By the Campbell-Baker-Hausdorff formula (1.1) we have $X + Y + \frac{1}{2}[X, Y] \in \log \Gamma$ for all $X, Y \in \log \Gamma$. It follows that $Z_1 + Z_2 = Z_1 + Z_2 + \frac{1}{2}[Z_1, Z_2] \in \Gamma_{\mathfrak{z}}$ for all $Z_1, Z_2 \in \Gamma_{\mathfrak{z}}$. Also, if $X = \pi(\tilde{X}), Y = \pi(\tilde{Y}) \in \Gamma_{\mathfrak{n}}$ with $\tilde{X}, \tilde{Y} \in \Gamma_{\mathfrak{n}}$, then $X + Y = \pi_{\mathfrak{n}}(\tilde{X} + \tilde{Y} + \frac{1}{2}[\tilde{X}, \tilde{Y}]) \in \Gamma_{\mathfrak{n}}$. Hence, the sets $\Gamma_{\mathfrak{z}}$ and $\Gamma_{\mathfrak{n}}$ are lattices.

PROPOSITION 1.9. *Let $(G, \mathbf{m})$ be a connected and simply-connected nonsingular 2-step nilpotent Lie group and $\Gamma \subset G$ a uniform subgroup. Furthermore, let $(Z_1, \ldots, Z_\ell, X_1, \ldots, X_{2n})$ be a basis of $\mathfrak{g}$ as in Proposition 1.8. Then $(Z_1, \ldots, Z_\ell)$ is a basis of $\Gamma_{\mathfrak{z}}$ and $(\pi_{\mathfrak{n}}(X_1), \ldots, \pi_{\mathfrak{n}}(X_{2n}))$ is a basis of $\Gamma_{\mathfrak{n}}$. In particular, $\Gamma_{\mathfrak{z}}$ and $\Gamma_{\mathfrak{n}}$ have full rank in $\mathfrak{z}$ and $\mathfrak{n}$ respectively.*

PROOF. By the last proposition and formula (1.1) we have

$$\begin{aligned} \Gamma &= \left\{ \exp(m_1 Z_1) \cdots \exp(m_\ell Z_\ell) \cdot \exp(m_{\ell+1} X_1) \cdots \exp(m_{\ell+2n} X_{2n}) \mid m \in \mathbb{Z}^{\ell+2n} \right\} \\ &= \left\{ \exp\left(\sum_{i=1}^{\ell} m_i Z_i + \sum_{i=1}^{2n} m_{\ell+i} X_i + \frac{1}{2} \sum_{1 \leqslant i < j \leqslant 2n} m_{\ell+i} m_{\ell+j} [X_i, X_j] \right) \mid m \in \mathbb{Z}^{\ell+2n} \right\} . \end{aligned}$$

It follows that

$$\log \Gamma = \left\{ \sum_{i=1}^{\ell} m_i Z_i + \sum_{i=1}^{2n} m_{\ell+i} X_i + \frac{1}{2} \sum_{1 \leqslant i < j \leqslant 2n} m_{\ell+i} m_{\ell+j} [X_i, X_j] \mid m \in \mathbb{Z}^{\ell+2n} \right\} .$$

By (1.12) we now have

$$\begin{aligned} \Gamma_{\mathfrak{z}} &= \log \Gamma \cap \mathfrak{z} = \left\{ \sum_{i=1}^{\ell} m_i Z_i \mid m \in \mathbb{Z}^{\ell} \right\} , \\ \Gamma_{\mathfrak{n}} &= \pi_{\mathfrak{n}} \log \Gamma = \left\{ \pi_{\mathfrak{n}} \left(\sum_{i=1}^{2n} m_i X_i \right) \mid m \in \mathbb{Z}^{2n} \right\} . \end{aligned}$$

Since $(Z_1, \ldots, Z_\ell)$ is a basis of $\mathfrak{z}$, it is a basis for $\Gamma_{\mathfrak{z}}$. The tuple $(\pi_{\mathfrak{n}}(X_1), \ldots, \pi_{\mathfrak{n}}(X_{2n}))$ is a basis of $\Gamma_{\mathfrak{n}}$ since $(Z_1, \ldots, Z_\ell, X_1, \ldots, X_{2n})$ is a basis of $\mathfrak{g}$ and $\mathfrak{z}$ is the kernel of $\pi_{\mathfrak{n}}$. □

The last proposition implies that

$$\begin{aligned} T_{\mathfrak{z}, \mathbf{m}_{\mathfrak{z}}} &:= (\Gamma_{\mathfrak{z}} \backslash \mathfrak{z}, \mathbf{m}_{\mathfrak{z}}) \ , \\ T_{\mathfrak{n}, \mathbf{m}_{\mathfrak{n}}} &:= (\Gamma_{\mathfrak{n}} \backslash \mathfrak{n}, \mathbf{m}_{\mathfrak{n}}) \ , \end{aligned} \tag{1.13}$$

are flat tori. We call $T_{\mathfrak{n}, \mathbf{m}_{\mathfrak{n}}}$ the *base torus* and $T_{\mathfrak{z}, \mathbf{m}_{\mathfrak{z}}}$ the *fibre torus* of $(\Gamma \backslash G, \mathbf{m})$. The justification for these names lies in the following proposition.

PROPOSITION 1.10 ([**Ebe94**, Proposition 5.5]). *The map*

$$\pi : (\Gamma\backslash G, \mathbf{m}) \to T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}$$
$$\Gamma \cdot \exp X \mapsto \Gamma_{\mathfrak{n}} + \pi_{\mathfrak{n}}(X)$$

is a Riemannian submersion whose fibres are totally geodesic imbedded flat tori isometric to $T_{\mathfrak{z},\mathbf{m}_{\mathfrak{z}}}$.

COROLLARY 1.11. *Let* $(\Gamma\backslash G, \mathbf{m})$ *be a compact Riemannian nilmanifold. Then*

$$\mathrm{Vol}(\Gamma\backslash G, \mathbf{m}) = \mathrm{Vol}\, T_{n,\mathbf{m}_{\mathfrak{n}}} \cdot \mathrm{Vol}\, T_{\mathfrak{z},\mathbf{m}_{\mathfrak{z}}}\,.$$

At last in this section, we describe the sectional curvature of the compact Riemannian nilmanifold $(\Gamma\backslash G, \mathbf{m})$. Since $(\Gamma\backslash G, \mathbf{m})$ is locally isometric to $(G, \mathbf{m})$ and the latter is homogeneous, it suffices to describe the sectional curvature $K^{\mathbf{m}}_{\sigma}(\Pi)$ at $e \in G$, the identity element, of the plane Π spanned by the linearly independent elements $X, Y \in \mathfrak{g} = T_eG$.

PROPOSITION 1.12 ([**Ebe94**, (2.4)]). *Let* $(G, \mathbf{m})$ *be a connected and simply connected 2-step nilpotent Lie group with a left invariant metric. Let* X, Y *be orthonormal elements of* $(\mathfrak{g}, \mathbf{m})$ *and* $\Pi := \mathrm{span}_{\mathbb{R}}\{X, Y\}$. *Then*

(a) $K^{\mathbf{m}}_{\sigma}(\Pi) = -\frac{3}{4}\|[X, Y]\|^2_{\mathbf{m}}$ *if* $X, Y \in \mathfrak{n}$.
(b) $K^{\mathbf{m}}_{\sigma}(\Pi) = \frac{1}{4}\|j(X)Y\|^2_{\mathbf{m}}$ *if* $X \in \mathfrak{z}$ *and* $Y \in \mathfrak{n}$.
(c) $K^{\mathbf{m}}_{\sigma}(\Pi) = 0$ *if* $X, Y \in \mathfrak{z}$.

Unfortunately, there is a sign error in [**Ebe94**, (2.4)(b)]. That there must be an error is clear as P. Eberlein himself mentions in the introduction of [**Ebe94**] that J. A. Wolf proved in [**Wol64**] that every nilpotent Lie group with a left invariant metric must admit both positive and negative sectional curvatures.

The Spectrum of compact Heisenberg-like Nilmanifolds. We consider the Laplacian Δ acting on complex valued C^∞-functions on a compact Riemannian nilmanifold $(\Gamma\backslash G, \mathbf{m})$. As is well known (see, e.g., [**BGM71**, Chapitre III]), the spectrum of Δ consists of nonnegative eigenvalues with finite multiplicites and its only accumulation point is $+\infty$.

The spectrum of all compact Riemannian 2-step nilmanifolds was computed by H. Pesce [**Pes93**]. We give a short review of these results and specialise to compact Heisenberg-like nilmanifolds afterwards. Let $(\Gamma\backslash G, \mathbf{m})$ be a compact Riemannian 2-step nilmanifold. Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. For every $\lambda \in \mathfrak{g}^*$ define an antisymmetric bilinear form $B_\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by $B_\lambda(X, Y) := \lambda([X, Y])$ and denote by $\mathfrak{g}_\lambda$ its kernel. Let furthermore u_λ be the endomorphism of $\mathfrak{g}$ defined by $B_\lambda(X, Y) = \mathbf{m}(X, u_\lambda(Y))$. Note that u_λ is skew-symmetric w.r.t. $\mathbf{m}$.

The group G acts on $\mathfrak{g}^*$ via the coadjoint map $\mathrm{Ad}_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$, $g \in G$:

$$(\mathrm{Ad}_g^* \lambda)(Y) = \lambda(\mathrm{Ad}_{g^{-1}} Y), \quad Y \in \mathfrak{g}, \lambda \in \mathfrak{g}^* .$$

Let $\mathscr{A}$ be a full set of representatives of the space of G-orbits in $\mathfrak{g}^*$ and define $\mathscr{A}(\Gamma) = \{\lambda \in \mathscr{A} \mid \lambda(\log \Gamma \cap \mathfrak{g}_\lambda) \subseteq \mathbb{Z}\}$. Now write $\mathscr{A}(\Gamma) = \mathscr{A}_1(\Gamma) \cup \mathscr{A}_2(\Gamma)$ where $\mathscr{A}_1(\Gamma) = \{\lambda \in \mathscr{A}(\Gamma) \mid \mathfrak{g}_\lambda = \mathfrak{g}\}$ and $\mathscr{A}_2(\Gamma) = \mathscr{A}(\Gamma) \backslash \mathscr{A}_1(\Gamma)$.

Finally, define $m_\lambda := 1$ for every $\lambda \in \mathscr{A}_1(\Gamma)$ and $m_\lambda := (\det \widetilde{B_\lambda})^{1/2}$ for $\lambda \in \mathscr{A}_2(\Gamma)$. Here, $\widetilde{B_\lambda}$ is the bilinear antisymmetric form induced by B_λ on the quotient $\mathfrak{g}_\lambda \backslash \mathfrak{g}$ and the determinant is taken w.r.t. a basis of the lattice $\mathscr{L}_\lambda$ which is the image of $\log \Gamma$ under the canonical projection $\mathfrak{g} \to \mathfrak{g}_\lambda \backslash \mathfrak{g}$.

THEOREM 1.13 (**[Pes93**, II Proposition 1]). *The spectrum $\Sigma(\Gamma\backslash G, \mathbf{m})$ of the Laplacian Δ of the compact Riemannian 2-step nilmanifold $(\Gamma\backslash G, \mathbf{m})$ is the union $\Sigma(\Gamma\backslash G, \mathbf{m}) = \bigcup_{\lambda \in \mathscr{A}(\Gamma)} \Sigma(\lambda, \mathbf{m})$, where*

- *if $\lambda \in \mathscr{A}_1(\Gamma)$: $\Sigma(\lambda, \mathbf{m}) = \{4\pi^2 \sum_{j=1}^n \lambda(X_j)^2\}$ for any orthonormal basis $(X_1, \ldots, X_n)$ of $\mathfrak{g}$ with respect to $\mathbf{m}$.*
- *if $\lambda \in \mathscr{A}_2(\Gamma)$: $\Sigma(\lambda, \mathbf{m}) = \{\mu(\lambda, p, \mathbf{m}) \mid p \in \mathbb{N}_0^k\}$ with*

$$\mu(\lambda, p, \mathbf{m}) = 4\pi^2 \sum_{j=1}^{\ell} \lambda(X_j)^2 + 2\pi \sum_{j=1}^{k} (2p_j + 1) d_j ,$$

where

(1) $\ell = \dim \mathfrak{g}_\lambda$, $k = (\dim \mathfrak{g} - \dim \mathfrak{g}_\lambda)/2 = (n - \ell)/2$,

(2) $(X_1, \ldots, X_\ell)$ is an orthonormal basis of $\mathfrak{g}_\lambda$ with respect to the restriction of $\mathbf{m}$ to $\mathfrak{g}_\lambda$,

(3) $\pm i d_j$, $1 \leqslant j \leqslant k$, are the eigenvalues of u_λ.

Furthermore, every eigenvalue σ has the multiplicity $\Sigma_\lambda m_\lambda$ where the sum is over all $\lambda \in \mathscr{A}$ such that $\sigma \in \Sigma(\lambda, \mathbf{m})$.

In case $(\Gamma\backslash G, \mathbf{m})$ is a nonsingular Heisenberg-like nilmanifold, the above theorem has a much simpler form. We introduce the dual lattices of $\Gamma_\mathfrak{z}$ and $\Gamma_\mathfrak{n}$ defined in (1.12):

$$\begin{aligned} \Gamma_\mathfrak{z}^* &:= \{\lambda \in \mathfrak{z} \mid \langle \lambda, X \rangle_\mathbf{m} \in \mathbb{Z} \,\forall X \in \Gamma_\mathfrak{z}\} , \\ \Gamma_\mathfrak{n}^* &:= \{\lambda \in \mathfrak{n} \mid \langle \lambda, X \rangle_\mathbf{m} \in \mathbb{Z} \,\forall X \in \Gamma_\mathfrak{n}\} . \end{aligned} \tag{1.14}$$

COROLLARY 1.14. *Let $(\Gamma\backslash G, \mathbf{m})$ be a compact Riemannian nonsingular Heisenberg-like nilmanifold. Furthermore, let $\ell := \dim \mathfrak{z}$ and $n := {}^{\dim \mathfrak{n}}/2$. Denote by $0 < c_1^\mathbf{m} \leqslant \ldots \leqslant c_n^\mathbf{m}$ the real numbers such hat $\pm i c_j^\mathbf{m} \|Z\|_\mathbf{m}$, $1 \leqslant j \leqslant n$, are the eigenvalues of $j(Z)$ for all $Z \in \mathfrak{z}$. Then the spectrum $\Sigma(\Gamma\backslash G, \mathbf{m})$ of the Laplacian Δ of $(\Gamma\backslash G, \mathbf{m})$ is the union $\Sigma(\Gamma\backslash G, \mathbf{m}) = \bigcup_{\lambda \in \Gamma_\mathfrak{n}^* \cup \Gamma_\mathfrak{z}^*} \Sigma(\lambda, \mathbf{m})$ where*

- *if* $\lambda \in \Gamma_{\mathfrak{n}}^*$*:* $\Sigma(\lambda, \mathbf{m}) = \{4\pi^2\|\lambda\|_{\mathbf{m}}^2\}$.
- *if* $\lambda \in \Gamma_{\mathfrak{z}}^*$*:* $\Sigma(\lambda, \mathbf{m}) = \{\mu(\lambda, p, \mathbf{m}) \mid p \in \mathbb{N}_0^n\}$ *with*

$$\mu(\lambda, p, \mathbf{m}) = 4\pi^2\|\lambda\|_{\mathbf{m}}^2 + 2\pi\|\lambda\|_{\mathbf{m}} \sum_{j=1}^{n} (2p_j + 1)c_j^{\mathbf{m}} .$$

Each eigenvalue $\sigma \in \Sigma(\lambda, \mathbf{m})$ *with* $\lambda \in \Gamma_{\mathfrak{n}}^*$ *has multiplicity* 1*, whereas each eigenvalue* $\sigma \in \Sigma(\lambda, \mathbf{m})$ *with* $\lambda \in \Gamma_{\mathfrak{z}}^*$ *has multiplicity* $\left(\prod_{j=1}^{n} c_j^{\mathbf{m}}\right) \|\lambda\|_{\mathbf{m}}^n \operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}$.

PROOF. Let $^\flat : \mathfrak{g} \to \mathfrak{g}^*$ be the metric isomorphism and $^\sharp : \mathfrak{g}^* \to \mathfrak{g}$ its inverse. Define $\Gamma_{\mathfrak{z}}' := \left(\Gamma_{\mathfrak{z}}^*\right)^\flat$ and $\Gamma_{\mathfrak{n}}' := (\Gamma_{\mathfrak{n}}^*)^\flat$. We will show that $\Gamma_{\mathfrak{n}}'$ and $\Gamma_{\mathfrak{z}}'\backslash\{0\}$ can be taken for $\mathscr{A}_1(\Gamma)$ and $\mathscr{A}_2(\Gamma)$, respectively.

By Remark 1.4.(i) the nonsingularity of $\mathfrak{g}$ is equivalent to $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{z}$ being surjective for all $X \in \mathfrak{n}\backslash\{0\}$. Let $\lambda \in \mathfrak{g}^*$ and $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid \lambda([X, Y]) = \lambda(\mathrm{ad}_X(Y)) = 0 \,\forall Y \in \mathfrak{g}\}$. Then $\mathfrak{g}_\lambda = \mathfrak{z}$ in case $\lambda_{|\mathfrak{z}} \neq 0$ and $\mathfrak{g}_\lambda = \mathfrak{g}$ if $\lambda_{|\mathfrak{z}} \equiv 0$. Since $\mathfrak{g}_\lambda$ is an ideal (every additive subspace of $\mathfrak{g}$ that contains the derived algebra $\mathfrak{g}' = \mathfrak{g}^{(2)} = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{z}$ is an ideal) the Ad_G^*-orbit through λ is exactly $\lambda + \mathfrak{g}_\lambda^\perp$, where $\mathfrak{g}_\lambda^\perp$ is the annihilator of $\mathfrak{g}_\lambda$ in $\mathfrak{g}^*$ (see **[CG90**, Theorem 3.2.3]).

If $\mathfrak{g}_\lambda = \mathfrak{g}$ then $\mathfrak{g}_\lambda^\perp = \{0\}$, i.e., the orbit through an element $\lambda \in \mathfrak{g}^*$ with $\lambda_{|\mathfrak{z}} \equiv 0$ contains only λ. Clearly, a parametrising set for all such λ which additionally satisfy $\lambda(\log\Gamma \cap \mathfrak{g}_\lambda) = \lambda(\log\Gamma) \subseteq \mathbb{Z}$ is $\Gamma_{\mathfrak{n}}'$.

Now let $\lambda \in \mathfrak{g}^*$ with $\lambda_{|\mathfrak{z}} \neq 0$. Then $\mathfrak{g}_\lambda = \mathfrak{z}$ and $\mathfrak{z}^\perp = \{\mu \in \mathfrak{g}^* \mid \mu(Z) = 0 \,\forall Z \in \mathfrak{z}\} = \mathfrak{n}^\flat$. Hence, the Ad_G^*-orbit through λ is $\lambda + \mathfrak{n}^\flat$. We can thus w.l.o.g. assume that $\lambda_{|\mathfrak{n}} \equiv 0$. The set of all $\lambda \in \mathfrak{g}^*$ such that $\lambda_{|\mathfrak{n}} \equiv 0$ and $\lambda(\log\Gamma \cap \mathfrak{g}_\lambda) = \lambda(\log\Gamma \cap \mathfrak{z}) = \lambda(\Gamma_{\mathfrak{z}}) \subseteq \mathbb{Z}$ is clearly $\Gamma_{\mathfrak{z}}'$. Hence, $\Gamma_{\mathfrak{z}}'\backslash\{0\} = \mathscr{A}_2(\Gamma)$ is a valid choice.

We will now prove with the help of Theorem 1.13 that the above eigenvalue and multiplicity formulas are correct. First, let $\lambda^\flat \in \Gamma_{\mathfrak{n}}'$ and $(X_1, \ldots, X_{\ell+2n})$ be an orthonormal basis of $\mathfrak{g}$. Then

$$4\pi^2 \sum_{j=1}^{\ell+2n} \lambda^\flat(X_j)^2 = 4\pi^2 \sum_{j=1}^{\ell+2n} \langle \lambda, X_j \rangle^2 = 4\pi^2\|\lambda\|_{\mathbf{m}}^2 .$$

Now, let $\lambda^\flat \in \Gamma_{\mathfrak{z}}'\backslash\{0\}$ and $(X_1, \ldots, X_\ell)$ be an orthonormal basis of $\mathfrak{z} = \mathfrak{g}_\lambda$. By the definition of $u_{\lambda^\flat}$ and (1.9) we have

$$\langle X, u_{\lambda^\flat}(Y) \rangle = \lambda^\flat([X, Y]) = \langle \lambda, [X, Y] \rangle = \langle j(\lambda)X, Y \rangle$$

for all $X, Y \in \mathfrak{n}$. It follows that $d_j = c_j^{\mathbf{m}}\|\lambda\|_{\mathbf{m}}$ for all $1 \leqslant j \leqslant n$ and that

$$\mu(\lambda^\flat, p, \mathbf{m}) = 4\pi^2 \sum_{j=1}^{\ell} \lambda^\flat(X_j)^2 + 2\pi \sum_{j=1}^{n} (2p_j + 1)d_j = 4\pi^2\|\lambda\|_{\mathbf{m}}^2 + 2\pi\|\lambda\|_{\mathbf{m}} \sum_{j=1}^{n} (2p_j + 1)c_j^{\mathbf{m}}$$

for all $p \in \mathbb{N}_0^n$.

It remains to prove the multiplicity formula for $\lambda^\flat \in \Gamma'_{\mathfrak{z}} \backslash \{0\}$. Since $\mathfrak{g}_\lambda = \mathfrak{z}$ we have $\mathfrak{g}_\lambda \backslash \mathfrak{g} \cong \mathfrak{n}$ and $\mathscr{L}_\lambda = \Gamma_{\mathfrak{n}}$. Let $(X_1, \dots, X_{2n})$ be a basis of $\Gamma_{\mathfrak{n}}$. Then

$$\begin{aligned} \det \widetilde{B_{\lambda^\flat}}^{1/2} &= \sqrt{\det \left(\lambda^\flat \left(\left[X_i, X_j\right]\right)\right)_{i,j}} = \sqrt{\det \left(\left\langle \lambda, \left[X_i, X_j\right]\right\rangle\right)_{i,j}} = \sqrt{\det \left(\left\langle j\left(\lambda\right) X_i, X_j\right\rangle\right)_{i,j}} \\ &= \sqrt{\det j\left(\lambda\right)} \operatorname{Vol} T_{\mathfrak{n}, \mathbf{m}_{\mathfrak{n}}} . \end{aligned}$$

□

REMARK 1.15. In light of Proposition 1.10, the eigenfunctions corresponding to the eigenvalues in $\Sigma(\lambda, \mathbf{m})$ with $\lambda \in \Gamma_{\mathfrak{n}}^*$ are precisely the lifts of eigenfunctions of the Laplace-Beltrami operator on the base torus $T_{\mathfrak{n}, \mathbf{m}_{\mathfrak{n}}}$.

2. The Manifold $\mathcal{P}_n$ of Symmetric and Positive Definite $n \times n$ Matrices

We we will throughout this thesis be concerned with the space of all inner products on a 2-step nilpotent Lie algebra and certain subspaces thereof. By a choice of basis, this space can be identified with the set of positive definite symmetric matrices. This can in turn be given the structure of a complete Riemannian homogeneous manifold, which we briefly describe below. We refer the reader to [**O'N83**, Ch. 9] for a good introduction to the general theory of homogeneous spaces and to the comprehensive [**Ter88**] for specifics about $\mathcal{P}_n$, $\mathcal{SP}_n$ and $\mathcal{P}_{2n}^*$.

NOTATION 1.16. For a quadratic matrix $A \in M(m; \mathbb{R})$ and a matrix $B \in M(m, n; \mathbb{R})$ we write

$$A[B] := {}^{\mathrm{t}}B \cdot A \cdot B \in M(n; \mathbb{R}) .$$

DEFINITION 1.17. For every $n \in \mathbb{N}$ we define

$$\begin{aligned} \mathcal{P}_n &:= \left\{Y \in M(n; \mathbb{R}) \mid {}^{\mathrm{t}}Y = Y, Y \text{ positive definite}\right\} , \\ \mathcal{SP}_n &:= \{Y \in \mathcal{P}_n \mid \det Y = 1\} \subset \mathcal{P}_n . \end{aligned}$$

Furthermore, for every $Y \in \mathcal{P}_{2n}$ we define

$$\mathcal{P}_{2n}^*(Y) := \left\{Y[S] \mid S \in \widetilde{\mathrm{Sp}}(2n; \mathbb{R})\right\} \subset \mathcal{P}_{2n} ,$$

where

$$\widetilde{\mathrm{Sp}}(2n; \mathbb{R}) := \left\{S \in \mathrm{GL}(2n; \mathbb{R}) \mid {}^{\mathrm{t}}SJS = \varepsilon(S)J, \varepsilon(S) = \pm 1\right\}$$

with J as in (1.3). In the special case $Y = \mathrm{Id}$, we use the abbreviation

$$\mathcal{P}_{2n}^* := \mathcal{P}_{2n}^*(\mathrm{Id}) .$$

There is a smooth $\mathrm{GL}(n;\mathbb{R})$-right action on $\mathcal{P}_n$ given by

$$\begin{aligned} \mathcal{P}_n \times \mathrm{GL}(n;\mathbb{R}) &\to \mathcal{P}_n \\ (Y,G) &\mapsto Y[G]\,. \end{aligned} \tag{1.15}$$

By the spectral theorem for symmetric matrices this action is transitive. Through restriction, we also obtain a smooth transitive $\mathrm{SL}(n;\mathbb{R})$-right action on $\mathcal{SP}_n$ and a smooth $\widetilde{\mathrm{Sp}}(2n;\mathbb{R})$-right action on $\mathcal{P}^*_{2n}(Y)$. The latter is transitive by definition of $\mathcal{P}^*_{2n}(Y)$.

REMARK 1.18. We have $\mathcal{P}^*_{2n} = \{\mathrm{Id}[S] \mid S \in \mathrm{Sp}(2n;\mathbb{R})\}$. Indeed, for if we let

$$K = \mathrm{diag}(1,\ldots,1,-1,\ldots,-1) \in \mathrm{GL}(2n;\mathbb{R})$$

with exactly n entries $+1$, then $K \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})\backslash\mathrm{Sp}(2n;\mathbb{R})$, ${}^{\mathrm{t}}K = K$, $K^2 = \mathrm{Id}$ and K maps $\mathrm{Sp}(2n;\mathbb{R})$ diffeomorphically onto $\widetilde{\mathrm{Sp}}(2n;\mathbb{R})\backslash\mathrm{Sp}(2n;\mathbb{R})$ via left multiplication. Hence for every $S' \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})$ with $S' \notin \mathrm{Sp}(2n;\mathbb{R})$, there is $S \in \mathrm{Sp}(2n;\mathbb{R})$ with $S' = KS$ and $\mathrm{Id}[S'] = \mathrm{Id}[KS] = \mathrm{Id}[K][S] = K^2[S] = \mathrm{Id}[S]$.

Thus, the manifold $\mathcal{P}^*_{2n}$ is a homogeneous manifold for the group $\mathrm{Sp}(2n;\mathbb{R})$. It is known as the Siegel upper half plane, see [**Ter88**, Chapter V]. One can show that $\mathcal{P}^*_{2n} = \mathcal{P}_{2n} \cap \mathrm{Sp}(2n;\mathbb{R})$ (see [**Ter88**, Chapter 5.1, Lemma 1(a)]).

The tangent spaces of $\mathcal{P}_n$, $\mathcal{SP}_n$ and $\mathcal{P}^*_{2n}$ at Id are given by

$$\begin{aligned} T_{\mathrm{Id}}\mathcal{P}_n &= \mathfrak{r}_n = \{X \in M(n;\mathbb{R}) \mid {}^{\mathrm{t}}X = X\}\,, \\ T_{\mathrm{Id}}\mathcal{SP}_n &= \mathfrak{q}_n = \{X \in M(n;\mathbb{R}) \mid {}^{\mathrm{t}}X = X, \operatorname{tr} X = 0\}\,, \\ T_{\mathrm{Id}}\mathcal{P}^*_{2n} &= \mathfrak{p}_{2n} = \mathfrak{q}_{2n} \cap \mathfrak{sp}(2n;\mathbb{R}) = \{X \in M(2n;\mathbb{R}) \mid {}^{\mathrm{t}}X = X, XJ + JX = 0\}\,. \end{aligned}$$

Here, $\mathfrak{sp}(2n;\mathbb{R})$ is the Lie algebra of $\mathrm{Sp}(2n;\mathbb{R})$. It is very well-known that the matrix exponential provides a diffeomorphism $\exp : \mathfrak{r}_n \to \mathcal{P}_n$ with $\exp(\mathfrak{q}_n) = \mathcal{SP}_n$ and $\exp(\mathfrak{p}_{2n}) = \mathcal{P}^*_{2n}$.

An inner product on $\mathfrak{r}_n$ is given by the trace form ([**Ter88**, (1.11) on p. 11]):

$$\mathfrak{r}_n \times \mathfrak{r}_n \ni (X,Y) \mapsto \operatorname{tr}(XY) \in \mathbb{R}\,.$$

We extend this inner product to a Riemannian metric on $\mathcal{P}_n$ by the $\mathrm{GL}(n;\mathbb{R})$-right action (1.15). Through restriction, we obtain Riemannian metrics on $\mathcal{SP}_n$ and $\mathcal{P}^*_{2n}$ that are invariant under the action of $\mathrm{SL}(n;\mathbb{R})$ and $\mathrm{Sp}(2n;\mathbb{R})$, respectively. The matrix exponential map $\exp : \mathfrak{r}_n \to \mathcal{P}_n$ is precisely the geodesic exponential map w.r.t. this metric (see [**Ter88**, Ch. IV, Theorem 1]). By this and the transitivity and isometry of the respective actions, we see that $\mathcal{SP}_n \subset \mathcal{P}_n$ and $\mathcal{P}^*_{2n} \subseteq \mathcal{SP}_{2n}$ are totally geodesic submanifolds. By the Hopf-Rinow Theorem, exp being a diffeomorphism also implies that $\mathcal{P}_n$ (resp. $\mathcal{SP}_n$ and $\mathcal{P}^*_{2n}$)

are complete Riemannian manifolds. By the same Theorem, this is equivalent to $\mathcal{P}_n$, $\mathcal{SP}_n$ and $\mathcal{P}^*_{2n}$ being complete metric spaces under the Riemannian distance.

Note that for any $r > 0$ one has $\mathcal{P}^*_{2n}(r \cdot \mathrm{Id}) = r \cdot \mathcal{P}^*_{2n}$. It follows that $\mathcal{P}^*_{2n}(r \cdot \mathrm{Id})$ is a totally geodesic submanifold of $\mathcal{P}_{2n}$ and a complete metric space under the Riemannian distance too.

3. Schwartz Functions, the Fourier Transform and the Poisson Summation Formula

Schwartz functions, their Fourier transform and the Poisson summation formula play important roles in this thesis. Following [**Gra14**, Section 2.2 and Section 3.2.3], we state all definitions, examples and theorems we use for reference.

For a point $x \in \mathbb{R}^n$, we denote by $\|x\| = \|x\|_2$ the Euclidean norm. A *multi-index* is an element of $\mathbb{N}_0^n$. The size of a multi-index $\alpha \in \mathbb{N}_0^n$ is defined as $|\alpha| := \alpha_1 + \alpha_2 + \ldots + \alpha_n$. For $x \in \mathbb{R}^n$ and α a multi-index, we define $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. It is a simple fact that for every $\alpha \in \mathbb{N}_0$ there exists a constant $c_{n,\alpha} > 0$ such that

$$|x^\alpha| \leqslant c_{n,\alpha} \|x\|^{|\alpha|} \tag{1.16}$$

for all $x \in \mathbb{R}^n$.

For a function $f \in C^\infty(\mathbb{R}^n)$ we denote by $\frac{\partial^m}{\partial x_j^m} f(x)$ the m-th derivative w.r.t. the j-th variable and for $\alpha \in \mathbb{N}_0$, we use the notation

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} f(x) := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f(x)\,.$$

DEFINITION 1.19. A complex valued function $f \in C^\infty(\mathbb{R}^n)$ is called a *Schwartz function* if for every pair of multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ there exists a $C_{\alpha,\beta} > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha \frac{\partial^{|\beta|}}{\partial x^\beta} f(x) \right| \leqslant C_{\alpha,\beta}\,. \tag{1.17}$$

The set of all Schwartz functions on $\mathbb{R}^n$ is denoted by $\mathscr{S}(\mathbb{R}^n)$.

REMARK 1.20. It follows directly from the definition of a Schwartz function that if $\alpha \in \mathbb{N}_0$, $f \in \mathscr{S}(\mathbb{R}^n)$ and P is a polynomial on $\mathbb{R}^n$, then $x \mapsto P(x) \cdot \frac{\partial^{|\alpha|}}{\partial x^\alpha} f(x) \in \mathscr{S}(\mathbb{R}^n)$.

REMARK 1.21. Note that Definition 1.19 is invariant under pullback by linear isomorphisms of $\mathbb{R}^n$. It follows that Definition 1.19 makes sense on any finite dimensional euclidean vector space.

EXAMPLE 1.22. The function $\mathbb{R}^n \ni x \mapsto e^{-\|x\|^2} \in \mathbb{R}$ lies in $\mathscr{S}(\mathbb{R}^n)$.

LEMMA 1.23 ([**Gra14**, cf. Remark 2.2.4]). *Let $f \in C^\infty(\mathbb{R}^n)$. Then f is in $\mathscr{S}(\mathbb{R}^n)$ if and only if for every $\alpha \in \mathbb{N}_0^n$ and every $N \in \mathbb{N}$ there exists $C_{\alpha,N}$ such that*

$$\left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} f(x) \right| \leqslant \frac{C_{\alpha,N}}{(1+\|x\|)^N} \tag{1.18}$$

for all $x \in \mathbb{R}^n$.

REMARK 1.24. The function

$$\mathbb{R}^n \ni x \mapsto \frac{1}{(1+\|x\|)^{n+1}} \in \mathbb{R}$$

is integrable. It follows from Lemma 1.23 that $\mathscr{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$.

DEFINITION 1.25. Let $f, g \in L^1(\mathbb{R}^n)$. The *convolution* of f and g is defined as the function

$$(f * g) : \mathbb{R}^n \ni x \mapsto \int_{\mathbb{R}^n} f(y) g(x-y) \mathrm{d}\mu(y) \in \mathbb{C}\,. \tag{1.19}$$

REMARK 1.26. An application of Fubini's Theorem shows that the right hand side of (1.19) is defined almost everywhere and that $f * g \in L^1(\mathbb{R}^n)$. Furthermore, the convolution as a map $L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ is associative.

PROPOSITION 1.27. *Let $f, g \in \mathscr{S}(\mathbb{R}^n)$. Then fg and $f * g$ are in $\mathscr{S}(\mathbb{R}^n)$. Moreover,*

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}(f * g) = \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} f \right) * g = f * \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} g \right)$$

for all multi-indices $\alpha \in \mathbb{N}_0^n$.

DEFINITION 1.28. Let $(V, \langle \cdot, \cdot \rangle)$ be a euclidean vector space. Given a Schwartz function $f \in \mathscr{S}(V)$ we define

$$F[f](\xi) := \widehat{f}(\xi) := \int_V e^{-2\pi i \langle x, \xi \rangle} f(x) \,\mathrm{dVol}_V(x)\,, \tag{1.20}$$

where dVol_V is the volume density determined by the Lebesgue measure and the inner prodct $\langle \cdot, \cdot \rangle$. We call $F[f] = \widehat{f}$ the *Fourier transform of f*.

Note that the Fourier transform $F[f] : V \to \mathbb{C}$ of a Schwartz function f always exists since $\mathscr{S}(V) \subset L^1(V, \mathrm{dVol}_V)$.

For a function $f : \mathbb{R}^n \to \mathbb{C}$, $y \in \mathbb{R}^n$ and $a > 0$ we define *translation, dilation* and *reflection* of f by

$$(\tau^y f)(x) := f(x-y)\,,$$
$$(\delta^a f)(x) := f(ax)\,,$$

$$\widetilde{f}(x) := f(-x)\,.$$

We also use the notation $f_a := a^{-n}(\delta^{1/a} f)$.

PROPOSITION 1.29 ([**Gra14**, Proposition 2.2.11]). *Given $f, g \in \mathscr{S}(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, $b \in \mathbb{C}$, $\alpha \in \mathbb{N}_0$ and $t > 0$, we have*

(i) $\widehat{f+g} = \widehat{f} + \widehat{g}$,
(ii) $\widehat{bf} = b\widehat{f}$,
(iii) $\widehat{\widetilde{f}} = \widetilde{\widehat{f}}$,
(iv) $\widehat{\overline{f}} = \overline{\widetilde{\widehat{f}}}$,
(v) $\widehat{\tau^y f}(\xi) = e^{-2\pi i \langle y, \xi\rangle} \widehat{f}(\xi)$,
(vi) $\left(x \mapsto e^{2\pi i \langle x, y\rangle} f(x)\right)\widehat{\ }(\xi) = \tau^y(\widehat{f})(\xi)$,
(vii) $\left(\delta^t f\right)\widehat{\ } = t^{-n}\delta^{1/t}\widehat{f} = (\widehat{f})_t$,
(viii) $\widehat{f} \in \mathscr{S}(\mathbb{R}^n)$,
(ix) $\left(x \mapsto \frac{\partial^{|\alpha|}}{\partial x^\alpha} f(x)\right)\widehat{\ }(\xi) = (2\pi i \xi)^\alpha \widehat{f}(\xi)$,
(x) $\left(x \mapsto (-2\pi i x)^\alpha f(x)\right)\widehat{\ }(\xi) = \frac{\partial^{|\alpha|}}{\partial \xi^\alpha} \widehat{f}(\xi)$,
(xi) $\widehat{f \cdot g} = \widehat{f} * \widehat{g}$,
(xii) $\widehat{f \circ A}(\xi) = \widehat{f}(A\xi)$ *for every* $A \in O(n; \mathbb{R})$.

EXAMPLE 1.30 (cf. [**Gra14**, Example 2.2.9]). Let $(V, \langle\cdot,\cdot\rangle)$ be a euclidean vector space and $f : V \ni x \mapsto e^{-\pi\|x\|^2} \in \mathbb{R}$. Then $\widehat{f} = f$.

DEFINITION 1.31. Given $f \in \mathscr{S}(\mathbb{R}^n)$, we define

$$\check{f} : \mathbb{R}^n \ni x \mapsto \widehat{f}(-x) \in \mathbb{C}\,.$$

Note that $\check{f} \in \mathscr{S}(\mathbb{R}^n)$ by Proposition 1.29*(viii)*. The map $\mathscr{S}(\mathbb{R}^n) \ni f \mapsto \check{f} \in \mathscr{S}(\mathbb{R}^n)$ is called *inverse Fourier Transform*.

THEOREM 1.32 ([**Gra14**, Theorem 2.2.14]). *Given $f \in \mathscr{S}(\mathbb{R}^n)$ we have*

$$(\widehat{f})\check{\ } = f = (\check{f})\widehat{\ }.$$

REMARK 1.33. By the last theorem and Proposition 1.29*(iii)* we have $\widehat{\widehat{f}} = f$ for any even Schwartz function f.

THEOREM 1.34. [**Gra14**, Theorem 3.2.8] *Let f be a continuous function on $\mathbb{R}^n$ which satisfies for some $C, \delta > 0$ and for all $x \in \mathbb{R}^n$*

$$|f(x)| \leqslant C(1 + \|x\|)^{-n-\delta}\,,$$

and whose Fourier transform $\widehat{f}$ restricted to $\mathbb{Z}^n$ satisfies

$$\sum_{m\in\mathbb{Z}^n} |\widehat{f}(m)| < \infty\,.$$

Then for all $x \in \mathbb{R}^n$ we have

$$\sum_{m\in\mathbb{Z}^n} \widehat{f}(m)e^{2\pi i\langle m,x\rangle_{std}} = \sum_{k\in\mathbb{Z}^n} f(x+k)\,,$$

and in particular

$$\sum_{m\in\mathbb{Z}^n} \widehat{f}(m) = \sum_{k\in\mathbb{Z}^n} f(k)\,.$$

COROLLARY 1.35. *Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space, $L \subset V$ a lattice of full rank and $f \in \mathscr{S}(V)$. Let L^* be the dual lattice of L, i.e.,*

$$L^* = \{v \in V \,|\, \langle v, m\rangle \in \mathbb{Z} \textit{ for all } m \in L\}\,.$$

Then we have

$$\sum_{m\in L} f(m) = \mathrm{Vol}\,(L^*\backslash V) \sum_{m\in L^*} \widehat{f}(m)\,.$$

PROOF. Identify V with $\mathbb{R}^n$ by a linear isometry and let $B \in \mathrm{GL}(n;\mathbb{R})$ be such that $L = B\cdot\mathbb{Z}^n$. Then $L^* = {}^{\mathrm{t}}B^{-1}\cdot\mathbb{Z}^n$. By Theorem 1.34 we now have

$$\sum_{m\in L} f(m) = \sum_{k\in\mathbb{Z}^n} f\circ B(k) = \sum_{m\in\mathbb{Z}^n} \widehat{f\circ B}(m)\,.$$

A simple application of integration by substitution shows that

$$\sum_{m\in\mathbb{Z}^n} \widehat{f\circ B}(m) = \left|\det B^{-1}\right| \sum_{m\in\mathbb{Z}^n} \widehat{f}\circ\left({}^{\mathrm{t}}B^{-1}\right)(m) = \mathrm{Vol}(L^*\backslash V) \sum_{m\in L^*} \widehat{f}(m)\,.$$

□

4. Parameter Dependent Integrals

We will often be in a situation in which we have to show the continuity, the differentiability or holomorphy of an integral depending on a parameter. We present here three standard theorems from [**Els05**] to avoid the reader having to browse through text books.

In the three following theorems, $(X,\mathfrak{A},\mu)$ denotes a measure space; i.e., $\mathfrak{A}$ is a σ-algebra over X and $\mu : \mathfrak{A} \to [0,\infty]$ a measure. Also, $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$ and $\mathscr{L}^1$ denotes the $\mathbb{K}$-valued integrable functions over X.

THEOREM 1.36 (Continuity of a Parameter Dependent Integral,
see [**Els05**, Kapitel IV,Satz 5.6]). *Let T be a metric space. Assume $f : T\times X \to \mathbb{K}$ possesses the following properties:*

(a) $f(t,\cdot) \in \mathscr{L}^1$ for every $t \in T$.

(b) $f(\cdot, x) \to \mathbb{K}$ *is continuous in* $t_0 \in T$ *for* μ*-almost all* $x \in X$.

(c) *There exists a neighborhood* U *of* t_0 *and a positive integrable function* $g : X \to \mathbb{R}$ *such that for all* $t \in U$ *one has:* $|f(t, \cdot)| \leqslant g$ μ*-almost everywhere.*

Then the function $F : T \to \mathbb{K}$ *defined by*

$$F(t) := \int_X f(t, x) \mathrm{d}\mu(x)$$

is continuous in $t_0 \in T$, *as is the function* $\Phi : T \ni t \mapsto f(t, \cdot) \in \mathscr{L}^1$.

THEOREM 1.37 (Differentiation under the Integral Sign, see [**Els05**, Kapitel IV, Sath 5.7]). *Let* $I \subset \mathbb{R}$ *be an interval and* $t_0 \in I$. *Assume* $f : I \times X \to \mathbb{K}$ *possesses the following properties:*

(a) $f(t, \cdot) \in \mathscr{L}^1$ *for every* $t \in T$.

(b) *The partial derivative* $\frac{\partial f}{\partial t}(t_0, x)$ *exists for all* $x \in X$.

(c) *There exists a neighorhood* U *of* t_0 *and a positive integrable function* $g : X \to \mathbb{R}$ *such that for all* $t \in U \cap I$, $t \neq t_0$, *one has*

$$\left| \frac{f(t, x) - f(t_0, x)}{t - t_0} \right| \leqslant g(x) \quad \mu\text{-almost everywhere} .$$

Then the function $F : I \to \mathbb{K}$ *defined by*

$$F(t) := \int_X f(t, x) \mathrm{d}\mu(x)$$

is (possibly one-sided) differentiable in t_0, $\frac{\partial f}{\partial t}(t_0, \cdot)$ *is integrable, and one has*

$$F'(t_0) = \int_X \frac{\partial f}{\partial t}(t_0, x) \mathrm{d}\mu(x) .$$

Supplement: The conclusion of the theorem still holds if one replaces assumptions (b), (c) by

(b)* *There exists* $\delta > 0$ *such that the partial derivative* $\frac{\partial f}{\partial t}(t, x)$, $x \in X$, *exists for all* $t \in U := (t_0 - \delta, t_0 + \delta) \cap I$.

(c)* *There exists a positive function* $g \in \mathscr{L}^1$ *such that for all* $t \in U$ *and* $x \in X$ *one has:*

$$\left| \frac{\partial f}{\partial t}(t, x) \right| \leqslant g(x) .$$

THEOREM 1.38 (Holomorphic Parameter Integral, see [**Els05**, Kapitel IV, Satz 5.8]). *Let* $G \subset \mathbb{C}$ *be open and assume* $f : G \times X \to \mathbb{C}$ *possesses the following properties:*

(a) $f(z, \cdot) \in \mathscr{L}^1$ *for all* $z \in G$.

(b) $f(\cdot, x) : G \to \mathbb{C}$ *is holomorphic for all* $x \in X$.

(c) *For every compact disc* $D \subset G$ *there exists a positive function* $g_D \in \mathscr{L}^1$ *such that for all* $z \in D$ *one has:* $|f(z, \cdot)| \leqslant g_D$ μ*-almost everywhere.*

Then the function $F : G \to \mathbb{C}$ defined by

$$F(z) := \int\limits_X f(z,x)\mathrm{d}\mu(x)$$

is holomorphic, $\frac{\partial^n f}{\partial z^n}(z,\cdot) \in \mathscr{L}^1$ *for all* $n \in \mathbb{N}_0$ *and one has*

$$F^{(n)}(z) := \int\limits_X \frac{\partial^n f}{\partial z^n}(z,x)\mathrm{d}\mu(x) \quad \textit{for all } z \in G\,.$$

5. Moduli Spaces

This section is devoted to the study of moduli spaces of compact nonsingular nilmanifolds. First, we introduce moduli spaces on a compact nonsingular nilmanifold for, respectively, all metrics, all Heisenberg-like metrics and all Heisenberg type metrics (Definition 1.43). We show that the numbers $c_1^{\mathbf{m}}, \ldots, c_j^{\mathbf{m}}$ from Definition 1.3*(ii)* are continuous functions on the moduli space of Heisenberg-like metrics (Proposition 1.40).

Then, we specialise to the case of the Heisenberg group H_n. Here, we introduce homeomorphic but simpler moduli spaces (Theorem 1.46). The main result (Theorem 1.63 and Corollary 1.64) of this section will be a characterisation of the compact sets of these moduli spaces.

The moduli space of all flat n-dimensional tori is the locally symmetric orbifold $\mathcal{P}_n / \operatorname{GL}(n;\mathbb{Z}) \simeq \operatorname{O}(n;\mathbb{R})\backslash \operatorname{GL}(n;\mathbb{R}) / \operatorname{GL}(n;\mathbb{Z})$. Here, a flat torus $T = \mathbb{R}^n/L$, where L is a lattice of full rank in $\mathbb{R}^n$, is identified with the congruence class $[Y] \in \mathcal{P}_n / \operatorname{GL}(n;\mathbb{Z})$, where Y is the Gram matrix of any $\mathbb{Z}$-basis of L. The flat torus T, or equivalently the lattice L, has two important invariants. One is its volume, which is the square root of the determinant of the Gram matrix Y. The other is the length of a shortest smoothly closed geodesic which is not homotopic to the identity. This length is the same as the length of a shortest nontrivial vector in L. We will denote the latter by $\sqrt{m(Y)}$. The classic Selection Theorem of Mahler (see Corollary 1.57) says that any set $M \subset \mathcal{P}_n / \operatorname{GL}(n;\mathbb{Z})$ for which there are constants $C_0, C_1 > 0$ such that $\det(Y) \leqslant C_1$ and $m(Y) \geqslant C_0$ for all $[Y] \in M$ has compact closure.

When we move from flat tori to compact Heisenberg manifolds, the moduli space essentially changes to $\mathcal{P}_{2n}/\widetilde{\operatorname{Sp}}(2n;\mathbb{Z}) \times (0,\infty)$ (see Theorem 1.46 for the exact definition). We know what the compact sets of the $(0,\infty)$ factor are. Now Example 1.59 shows that the existence of bounds for $\det(\cdot)$ and $m(\cdot)$ on a set $M \subset P_{2n}/\widetilde{\operatorname{Sp}}(2n;\mathbb{R})$ is in general not enough to ensure relative compactness of M. This asks for additional invariants. The action of the group $\widetilde{Sp}(2n;\mathbb{Z})$ on $Y \in \mathcal{P}_{2n}$ leaves invariant the eigenvalues of $Y^{-1}J$, where J is the matrix representation of the standard symplectic form on $\mathbb{R}^{2n}$ (see (1.3)).

Accordingly, Theorem 1.63 states that any set $M \subset \mathcal{P}_{2n}/\widetilde{\mathrm{Sp}}(2n;\mathbb{R})$ for which there are uniform bounds for $det(Y)$, $m(Y)$ and the eigenvalues of $Y^{-1}J$ for every $[Y] \in M$, is relatively compact.

DEFINITION AND REMARKS 1.39. Let G be a connected and simply-connected nonsingular 2-step nilpotent Lie group and $\mathfrak{g}$ its Lie algebra.

(i) We define the following three sets:

$$\mathscr{M}(G) := \{\mathbf{m} \mid \mathbf{m} \text{ is an inner product on } \mathfrak{g}\} ,$$
$$\mathscr{M}^{HL}(G) := \{\mathbf{m} \in \mathscr{M}(G) \mid (\mathfrak{g}, \mathbf{m}) \text{ is a Heisenberg-like Lie algebra}\} ,$$
$$\mathscr{M}^{HT}(G) := \{\mathbf{m} \in \mathscr{M}(G) \mid (\mathfrak{g}, \mathbf{m}) \text{ is an H-type Lie algebra}\} .$$

Note that we have the two inclusions $\mathscr{M}^{HT}(G) \subseteq \mathscr{M}^{HL}(G) \subseteq \mathscr{M}(G)$ by Definition 1.3.

(ii) By our convention made in Section 1 we regard every element in $\mathscr{M}(G)$ as a left invariant Riemannian metric on G and as a Riemannian metric on every compact nilmanifold $\Gamma\backslash G$, where $\Gamma \subset G$ is a uniform subgroup.

(iii) For applications of Theorem 1.36 with $T = \mathscr{M}(G)$ we need a metric on $\mathscr{M}(G)$. We turn $\mathscr{M}(G)$ into a complete metric space as follows. Fix a basis $(X_1, \ldots, X_{\ell+2n})$ of $\mathfrak{g}$ and define a diffeomorphism $\mathscr{M}(G) \to \mathcal{P}_{\ell+2n}$ by $\mathbf{m} \mapsto (\mathbf{m}(X_i, X_j))_{i,j=1,\ldots,\ell+2n}$. Here, $\mathcal{P}_{\ell+2n}$ is the space of symmetric positive definite matrices of dimension $\ell + 2n$, see Definition 1.17. Now define the metric on $\mathscr{M}(G)$ to be the pullback of the complete metric on $\mathcal{P}_{\ell+2n}$ (see Section 2) under this diffeomorphism.

PROPOSITION 1.40. *Let G be a connected and simply connected nonsingular 2-step nilpotent Lie group with Lie algebra $\mathfrak{g}$. Define $\ell := \dim \mathfrak{z}$ and $n := {}^{(\dim \mathfrak{g}) - \ell}/{}_2$.*

(i) $\mathscr{M}^{HL}(G)$ and $\mathscr{M}^{HT}(G)$ are closed in $\mathscr{M}(G)$. In particular, they are complete metric spaces.

(ii) For $\mathbf{m} \in \mathscr{M}^{HL}(G)$ denote by $0 < c_1^{\mathbf{m}} \leqslant \ldots \leqslant c_n^{\mathbf{m}}$ the real numbers associated with $\mathbf{m}$ such that $\pm i c_j^{\mathbf{m}} \|Z\|_{\mathbf{m}}$, $j = 1, \ldots, n$, are the eigenvalues of $j_{\mathbf{m}}(Z)$, where $j_{\mathbf{m}} : \mathfrak{z} \to \mathfrak{so}(\mathfrak{n})$ is the structure map. Then each $c_j^{\mathbf{m}}$ is continuous on $\mathscr{M}^{HL}(G)$.

PROOF. We want to work with the structure map $j_{\mathbf{m}} : \mathfrak{z} \to \mathfrak{so}(\mathfrak{n})$ as a function of the metric $\mathbf{m}$. Unfortunately, the subspace $\mathfrak{n} \subset \mathfrak{g}$ depends itself on $\mathbf{m}$ and so it is not clear what the range of this function should be. We remedy this by defining a map $j'_{\mathbf{m}} : \mathfrak{z} \to \mathfrak{gl}(\mathfrak{g})$ by

$$\langle j'_{\mathbf{m}}(Z)X, Y\rangle_{\mathbf{m}} = \langle Z, [X, Y]\rangle_{\mathbf{m}}$$

for all $X, Y \in \mathfrak{g}$. Clearly, we have $j'_{\mathbf{m}}(Z)X = j_{\mathbf{m}}(Z)X$ for all $Z \in \mathfrak{z}$ and $X \in \mathfrak{n}$. Additionally, we have $j'_{\mathbf{m}}(Z)X = 0$ for all $Z, X \in \mathfrak{z}$. Just as $j_{\mathbf{m}}$, $j'_{\mathbf{m}}$ is skew-symmetric w.r.t. $\mathbf{m}$. Note that as an element of $\mathrm{Hom}(\mathfrak{z}, \mathfrak{gl}(\mathfrak{g}))$, $j'_{\mathbf{m}}$ depends continuously on $\mathbf{m} \in \mathscr{M}(G)$. Let $0 < \theta_1(\mathbf{m}, Z) \leqslant \ldots \leqslant \theta_n(\mathbf{m}, Z)$ be the real numbers such that $\pm i\theta_j(\mathbf{m}, Z)$, $j = 1, \ldots, n$, are the nonzero eigenvalues of $j'_{\mathbf{m}}(Z)$. By [**Zed65**, Theorem 1], each of the maps $\mathscr{M}(G) \times \mathfrak{z} \ni (\mathbf{m}, Z) \mapsto \theta_j(\mathbf{m}, Z) \in (0, \infty)$ is continuous. Since $\mathfrak{z}$ and $\mathfrak{n}$ are invariant subspaces of $j'_{\mathbf{m}}(Z)$ for all $Z \in \mathfrak{z}$ and $\mathbf{m} \in \mathscr{M}(G)$, $\pm i\theta_j(\mathbf{m}, Z)$, $j = 1, \ldots, n$, are precisely the eigenvalues of $j_{\mathbf{m}}(Z)$ for all $Z \in \mathfrak{z}$ and $\mathbf{m} \in \mathscr{M}(G)$.

Let $(\mathbf{m}_k)_{k \in \mathbb{N}}$ be a sequence in $\mathscr{M}^{HL}(G)$ that converges in $\mathscr{M}(G)$ and $\mathbf{m} := \lim_{k\to\infty} \mathbf{m}_k$. Furthermore, let $0 < c_1^{\mathbf{m}_k} \leqslant \ldots \leqslant c_n^{\mathbf{m}_k}$ be the real numbers such that

$$\theta_j(\mathbf{m}_k, Z) = c_j^{\mathbf{m}_k} \|Z\|_{\mathbf{m}_k}$$

for all $Z \in \mathfrak{z}$ and $k \in \mathbb{N}$. Pick a $Z_0 \in \mathfrak{z} \backslash \{0\}$ and define

$$c_j := \lim_{k\to\infty} \frac{\theta_j(\mathbf{m}_k, Z_0)}{\|Z_0\|_{\mathbf{m}_k}}$$

for all $j = 1, \ldots, n$. We want to show that $\theta_j(\mathbf{m}, Z) = c_j \|Z\|_{\mathbf{m}}$ for all $Z \in \mathfrak{z}$ and $j = 1, \ldots, n$. Let $p \in \{1, 2, \ldots, n\}$. Since $\mathbf{m}_k \in \mathscr{M}^{HL}(G)$ we have

$$\frac{\theta_p(\mathbf{m}_k, Z_1)}{\|Z_1\|_{\mathbf{m}_k}} = c_p^{\mathbf{m}_k} = \frac{\theta_p(\mathbf{m}_k, Z_0)}{\|Z_0\|_{\mathbf{m}_k}}$$

for all $Z_1 \in \mathfrak{z} \backslash \{0\}, k \in \mathbb{N}$, and by taking the limit $k \to \infty$ we obtain $\theta_p(\mathbf{m}, Z_1) = c_p \|Z_1\|_{\mathbf{m}}$. Thus, we have

$$\theta_j(\mathbf{m}, Z) = c_j \|Z\|_{\mathbf{m}}$$

for all $j = 1, \ldots, n$ and $Z \in \mathfrak{z}$. This proves that $\mathscr{M}^{HL}(G)$ is a closed subspace of $\mathscr{M}(G)$.

Note that $\mathscr{M}(G) \times \mathfrak{z} \ni (\mathbf{m}, Z) \mapsto \|Z\|_{\mathbf{m}} \in [0, \infty)$ is continuous. It follows that

$$\mathscr{M}^{HL}(G) \ni \mathbf{m} \mapsto c_j^{\mathbf{m}} = \frac{\theta_j(\mathbf{m}, Z)}{\|Z\|_{\mathbf{m}}} \in (0, \infty)\,, \quad Z \in \mathfrak{z} \backslash \{0\}\,,$$

is continuous. This proves *(ii)*. To complete the proof of *(i)*, we note that $\mathscr{M}^{HT}(G) = \left\{ \mathbf{m} \in \mathscr{M}^{HL}(G) \mid c_j^{\mathbf{m}} = 1 \text{ for all } j = 1, \ldots, n \right\}$. □

REMARK 1.41. Note that $\mathscr{M}^{HL}(G)$ inherits local compactness from $\mathscr{M}(G)$. Even better, every closed ball in $\mathscr{M}^{HL}(G)$ is compact by the last Proposition. This property is called *properness*, i.e., $\mathscr{M}^{HL}(G)$ is a proper metric space and so is $\mathscr{M}^{HT}(G)$.

By the simply connectedness of G and the naturality of the exponential map, the map

$$\begin{aligned} \mathrm{Aut}(G) &\to \mathrm{Aut}(\mathfrak{g}) \\ \Phi &\mapsto \varphi = \mathrm{d}\Phi_e \end{aligned}$$

is a Lie group isomorphism, which is why we will identify an autormorphism of G with its differential at the identity henceforth.

The group $\mathrm{Aut}(\mathfrak{g})$ acts on $\mathscr{M}(G)$ via pullback, i.e.,

$$\begin{aligned}\mathscr{M}(G) \times \mathrm{Aut}(\mathfrak{g}) &\to \mathscr{M}(G)\\ (\mathbf{m}, \varphi) &\mapsto \varphi^* \mathbf{m}\,.\end{aligned}$$

This action is smooth. Indeed, by a choice of basis of $\mathfrak{g}$ this action corresponds to a restriction of the $\mathrm{GL}(\dim \mathfrak{g}; \mathbb{R})$-right action on $\mathcal{P}_{\dim \mathfrak{g}}$ defined in Section 2.

Note that the group $\mathrm{Aut}(G)$ acts on the space of left-invariant Riemannian metrics on G via pullback. This action corresponds to the above action of $\mathrm{Aut}(\mathfrak{g})$ on $\mathscr{M}(G)$ via the identification of $\mathrm{Aut}(G)$ with $\mathrm{Aut}(\mathfrak{g})$ and the identification made in Remark 1.39.*(ii)*.

Let $\mathrm{Inn}(\mathfrak{g})$ be the subgroup of inner automorphisms of $\mathfrak{g}$.

PROPOSITION 1.42. *Let G be a connected and simply connected nonsingular 2-step nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $\Gamma \subset G$ a uniform subgroup. Furthermore, let $\mathbf{m}_1, \mathbf{m}_2 \in \mathscr{M}(G)$ and define $C(\log \Gamma, \mathfrak{g}) := \{\varphi \in \mathrm{Aut}(\mathfrak{g}) \mid \varphi(\log \Gamma) = \log \Gamma\}$. Then the compact nilmanifolds $(\Gamma \backslash G, \mathbf{m}_1)$ and $(\Gamma \backslash G, \mathbf{m}_2)$ are isometric if and only if there exists $\varphi \in \mathrm{Inn}(\mathfrak{g}) \cdot C(\log \Gamma, \mathfrak{g})$ such that $\mathbf{m}_1 = \varphi^* \mathbf{m}_2$.*

PROOF. By the definition of 2-step nilpotency, the map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}_X Y = [X, Y]$ is nilpotent for all $X \in \mathfrak{g}$. This implies that the eigenvalues of every ad_X are zero, and so, in particular, real. Thus the Lie algebra $\mathfrak{g}$ has only real roots. The statement of the proposition now follows from [**GW88**, Theorem 5.4]. ☐

DEFINITION 1.43. Let G be a connected and simply connected nonsingular 2-step nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $\Gamma \subset G$ a uniform subgroup. Define an equivalence relation $\sim$ on $\mathscr{M}(G)$ by

$$\mathbf{m}_1 \sim \mathbf{m}_2 \text{ if and only if } \exists\, \varphi \in \mathrm{Inn}(\mathfrak{g}) \cdot C(\log \Gamma, \mathfrak{g}) \text{ with } \mathbf{m}_1 = \varphi^* \mathbf{m}_2\,.$$

With this equivalence relation, define the following three quotients:

$$\begin{aligned}\mathscr{M}(\Gamma, G) &:= \mathscr{M}(G)/\sim\,,\\ \mathscr{M}^{HL}(\Gamma, G) &:= \mathscr{M}^{HL}(G)/\sim\,,\\ \mathscr{M}^{HT}(\Gamma, G) &:= \mathscr{M}^{HT}(G)/\sim\ .\end{aligned}$$

By Proposition 1.42 it is clear that $\mathscr{M}(\Gamma, G)$ is a moduli space for $\Gamma \backslash G$, i.e, $(\Gamma \backslash G, \mathbf{m}_1)$ is isometric to $(\Gamma \backslash G, \mathbf{m}_2)$ if and only if $[\mathbf{m}_1] = [\mathbf{m}_2] \in \mathscr{M}(\Gamma, G)$. The subspaces $\mathscr{M}^{HT}(\Gamma, G) \subset \mathscr{M}^{HL}(\Gamma, G) \subseteq \mathscr{M}(\Gamma, G)$ are precisely the spaces of (equivalence classes of) metrics $[\mathbf{m}]$ such that $(\Gamma \backslash G, \mathbf{m})$ is a Heisenberg type or Heisenberg-like nilmanifold, respectively.

We equip $\mathscr{M}(\Gamma, G)$ with the quotient topology. The subspaces $\mathscr{M}^{HL}(\Gamma, G)$ and $\mathscr{M}^{HT}(\Gamma, G)$ are closed since $\mathscr{M}^{HL}(G)$ and $\mathscr{M}^{HT}(G)$ are saturated sets w.r.t. the canonical projection $\pi : \mathscr{M}(G) \to \mathscr{M}(\Gamma, G)$, i.e., $\mathscr{M}^{HL}(G) = \pi^{-1}(\pi(\mathscr{M}^{HL}(G)))$ and likewise for $\mathscr{M}^{HT}(G)$.

We will now specialise to the case $G = H_n$, the $(2n+1)$-dimensional Heisenberg group. The contents of what follows up to and including Theorem 1.46 are taken from **[GW86]**. Recall the definitions made in Example 1.2. For $x, y \in \mathbb{R}^n$, $s \in \mathbb{R}$ we let

$$\gamma(x,y,s) = \begin{pmatrix} 1 & {}^{\mathrm{t}}x & s \\ 0 & \mathrm{Id}_n & y \\ 0 & 0 & 1 \end{pmatrix}, \quad X(x,y,s) = \begin{pmatrix} 0 & {}^{\mathrm{t}}x & s \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

The $(2n+1)$-dimensional Heisenberg group H_n is $H_n = \{\gamma(x,y,s) \mid x, y \in \mathbb{R}^n, s \in \mathbb{R}\}$ with the group structure that it inherits as a subset of $\mathrm{GL}(n+2;\mathbb{R})$. Its Lie algebra is $\mathfrak{h}_n = \{X(x,y,s) \mid x, y \in \mathbb{R}^n, s \in \mathbb{R}\}$. The standard basis $\mathfrak{B}_n$ of $\mathfrak{h}_n$ is

$$\mathfrak{B}_n = (X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z)$$

with

$$\begin{aligned} X_j &= X(e_j, 0, 0) \text{ for all } 1 \leqslant j \leqslant n\,, \\ Y_j &= X(0, e_j, 0) \text{ for all } 1 \leqslant j \leqslant n\,, \\ Z &= X(0,0,1)\,, \end{aligned} \tag{1.21}$$

where $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$. Multiplication and inversion in H_n obey the rules

$$\gamma(x,y,s) \cdot \gamma(x',y',s') = \gamma(x+x', y+y', s+s'+\langle x, y' \rangle)\,,$$
$$\gamma(x,y,s)^{-1} = \gamma\left(-x, -y, -s + \langle x, y \rangle\right)\,.$$

It follows that commutators in H_n and $\mathfrak{h}_n$ are given by

$$\begin{aligned} [\gamma(x,y,s), \gamma(x',y',s')] &= \gamma\left(0, 0, A((x,y),(x',y'))\right)\,, \\ [X(x,y,s), X(x',y',s')] &= X\left(0, 0, A((x,y),(x',y'))\right)\,, \end{aligned} \tag{1.22}$$

where A is as in Example 1.2.

The centre $\mathfrak{z} = \mathfrak{z}_n$ of $\mathfrak{h}_n$ is $\mathfrak{z}_n = \{X(0,0,s) \mid s \in \mathbb{R}\}$. We identify the subspace $\{X(x,y,0) \mid (x,y) \in \mathbb{R}^{2n}\}$ with $\mathbb{R}^{2n}$. Under this identification $\mathfrak{h}_n$ is the direct sum $\mathfrak{h}_n = \mathbb{R}^{2n} \bigoplus \mathfrak{z}$ and $(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ is the standard basis of $\mathbb{R}^{2n}$. By (1.22) we have for all $X, Y \in \mathbb{R}^{2n}$

$$[X, Y] = A(X,Y) Z\,.$$

We now turn to the automorphisms of $\mathfrak{h}_n$. We will identify an automorphism φ with its matrix representation relative to the basis $\mathfrak{B}_n$. Recall from Definition 1.17: $\widetilde{\mathrm{Sp}}(2n;\mathbb{R}) =$

$\{\beta \in \mathrm{GL}(2n;\mathbb{R}) \mid {}^{\mathrm{t}}\beta J \beta = \epsilon(\beta) J,\, \varepsilon(\beta) = \pm 1\}$, with J as in (1.3). We imbed $\widetilde{\mathrm{Sp}}(2n;\mathbb{R})$ into $\mathrm{GL}(2n+1;\mathbb{R})$ via

$$\widetilde{\mathrm{Sp}}(2n;\mathbb{R}) \ni \beta \mapsto \begin{pmatrix} \beta & 0 \\ 0 & \epsilon(\beta) \end{pmatrix} \in \mathrm{GL}(2n+1;\mathbb{R})\,. \tag{1.23}$$

For $a \in \mathbb{R}\backslash\{0\}$ and $w \in \mathbb{R}^{2n}$, let

$$\alpha(a,w) := \begin{pmatrix} a \cdot I_{2n} & 0 \\ w^t & a^2 \end{pmatrix} \in \mathrm{GL}(2n+1;\mathbb{R})\,. \tag{1.24}$$

Simple calculations show that the group $\mathrm{Aut}(\mathfrak{h}_n)$ consists of all products of the form $\alpha(a,w)\cdot\beta$, $a \in \mathbb{R}\backslash\{0\}, w \in \mathbb{R}^{2n}, \beta \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})$. The inner automorphisms are those for which $a = 1$ and $\beta = \mathrm{Id}$.

We introduce a set of distinguished uniform subgroups of H_n. Let

$$\mathcal{D}_n := \{r = (r_1,\ldots,r_n) \in \mathbb{N}^n \mid \forall i \in \{1,\ldots,n-1\} : r_i | r_{i+1}\}\,. \tag{1.25}$$

For an $r \in \mathcal{D}_n$ define the matrix

$$\delta_r := \mathrm{diag}\,(r_1,\ldots,r_n,1,\ldots,1) \tag{1.26}$$

and the uniform subgroup $\Gamma^r \subset H_n$ by

$$\Gamma^r := \left\{\gamma(x,y,s) \mid \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \delta_r \cdot \mathbb{Z}^{2n}, s \in \mathbb{Z}\right\}\,. \tag{1.27}$$

THEOREM 1.44 ([**GW86**, Theorem 2.4]). *The subgroups Γ^r defined in* (1.27) *classify the uniform discrete subgroups of H_n up to automorphism; that is, if $\Gamma \subset H_n$ is any uniform subgroup of H_n, then there exists a unique $r \in \mathcal{D}_n$ for which some automorphism of H_n maps Γ to Γ^r. Moreover, for $r,s \in \mathcal{D}_n$, Γ^r and Γ^s are isomorphic if and only if $r = s$.*

We address the Riemannian structure of Heisenberg manifolds. Via the basis $\mathfrak{B}_n$, we identify the space $\mathscr{M}(H_n)$ of inner products on $\mathfrak{h}_n$ with $\mathcal{P}_{2n+1}$. In case that for a given metric $\mathbf{m} \in \mathcal{P}_{2n+1}$ the subspaces $\mathfrak{z} \subset \mathfrak{h}_n$ and $\mathbb{R}^{2n} \subset \mathfrak{h}_n$ are orthogonal, the metric takes the form

$$\mathbf{m} = \begin{pmatrix} h & 0 \\ 0 & g \end{pmatrix} \in \mathcal{P}_{2n} \times (0,\infty)\,. \tag{1.28}$$

We call such a metric *normalised* and write $\mathbf{m} = (h,g)$. Note that in this case $\mathfrak{n} = \mathbb{R}^{2n}$, $\mathbf{m}_{\mathfrak{n}} = h$ and $\mathbf{m}_{\mathfrak{z}} = g$.

PROPOSITION 1.45. *For every metric $\mathbf{m} \in \mathcal{P}_{2n+1}$ there exists exactly one $\varphi \in \mathrm{Inn}(\mathfrak{h}_n)$ such that $\varphi^*\mathbf{m}$ is a normalised metric. Moreover, φ depends continuously on $\mathbf{m}$.*

PROOF. Let $\mathbf{m} \in \mathcal{P}_{2n+1}$ and write

$$\mathbf{m} = \begin{pmatrix} h & w \\ {}^{\mathrm{t}}w & g \end{pmatrix}$$

with $h \in \mathcal{P}_{2n}$, $w \in \mathbb{R}^{2n}$ and $g \in (0, \infty)$. With $\alpha(\cdot, \cdot)$ as in (1.24) we have

$$\begin{aligned}(\alpha(1, -g^{-1}w))^* \mathbf{m} &= \begin{pmatrix} \mathrm{Id}_{2n} & -g^{-1}w \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} h & w \\ {}^{\mathrm{t}}w & g \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Id}_{2n} & 0 \\ -g^{-1}\,{}^{\mathrm{t}}w & 1 \end{pmatrix} \\ &= \begin{pmatrix} h - g^{-1}w \cdot {}^{\mathrm{t}}w & g(-g^{-1})w + w \\ {}^{\mathrm{t}}w & g \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Id}_{2n} & 0 \\ -g^{-1}\,{}^{\mathrm{t}}w & 1 \end{pmatrix} \\ &= \begin{pmatrix} h - g^{-1}w \cdot {}^{\mathrm{t}}w & 0 \\ 0 & g \end{pmatrix}.\end{aligned}$$

□

We call a Riemannian Heisenberg manifold $(\Gamma^r \backslash H_n, \mathbf{m})$, where $\mathbf{m} = (h, g)$ is a normalised metric, a *normalised Heisenberg manifold*.

If $\mathbf{m} = (h, g)$ is a normalised metric and $\alpha(a, w) \cdot \beta \in \mathrm{Aut}(\mathfrak{h}_n)$, then $(\alpha(a, w) \cdot \beta)^* \mathbf{m}$ is a normalised metric if and only if $w = 0$. Furthermore, for $r \in \mathcal{D}_n$ we have $\alpha(a, 0) \cdot \beta \log \Gamma^r = \log \Gamma^r$ if and only if $a = 1$ and $\beta \in \delta_r \,\mathrm{GL}(2n; \mathbb{Z}) \delta_r^{-1}$. We therefore define

$$G_r := \delta_r \,\mathrm{GL}(2n; \mathbb{Z}) \delta_r^{-1}, \qquad \Pi_r := G_r \cap \widetilde{\mathrm{Sp}}(2n; \mathbb{R}). \tag{1.29}$$

Combining the above with Proposition 1.42, Theorem 1.44 and Proposition 1.45 we obtain

THEOREM 1.46 (cmp. [**GW86**]). *Every compact Riemannian Heisenberg manifold is isometric to a normalised Heisenberg manifold. Moreover, two normalised Heisenberg manifolds* $(\Gamma^r \backslash H_n, \mathbf{m})$ *and* $\left(\Gamma^{r'} \backslash H_n, \mathbf{m}'\right)$ *are isometric iff* $r = r'$ *and* $\mathbf{m}' = \beta^* \mathbf{m}$, *where* $\beta \in \Pi_r = \widetilde{\mathrm{Sp}}(2n; \mathbb{R}) \cap \delta_r \,\mathrm{GL}(2n; \mathbb{Z}) \delta_r^{-1}$. *They are homeomorphic iff* $r = r'$. *Accordingly, the set*

$$\mathcal{M}_n := \bigcup_{r \in \mathcal{D}_n} \mathcal{M}_n^r, \qquad \mathcal{M}_n^r := (\mathcal{P}_{2n} \times (0, \infty)) / \Pi_r$$

parametrises the isometry classes of compact Riemannian Heisenberg manifolds. For every $n \in \mathbb{N}$ *and* $r \in \mathcal{D}_n$ *the map*

$$\begin{aligned} \mathcal{M}_n^r &\to \mathscr{M}(\Gamma^r, H_n) \\ [\mathbf{m}] &\mapsto [\overline{\mathbf{m}}], \end{aligned}$$

is a homeomorphism. Here, $\overline{\mathbf{m}}$ *is the inner product on* $\mathfrak{h}_n$ *whose matrix representation w.r.t. the basis* $\mathfrak{B}_n$ *is* $\mathbf{m}$.

Let $Y \in \mathcal{P}_{2n}$. Note that the map $Y^{-1}J$ is skew-symmetric w.r.t. the inner product defined by Y. It follows that the eigenvalues of $Y^{-1}J$ are purely imaginary and come in complex conjugate pairs.

DEFINITION AND REMARKS 1.47.

(i) For every $n \in \mathbb{N}$ we define n functions $d_1, \dots, d_n : \mathcal{P}_{2n} \to (0, \infty)$ such that $\pm i d_k(Y)$, $k = 1, \dots, n$, are the eigenvalues of $Y^{-1}J$ and $d_1 \leqslant d_2 \leqslant \dots \leqslant d_n$.

(ii) Note that the d_k are continuous (see, e.g., **[Zed65**, Theorem 1]).

(iii) The functions $d_1, \dots, d_n : \mathcal{P}_{2n} \to (0, \infty)$ are invariant under the action of $\widetilde{\mathrm{Sp}}(2n;\mathbb{R})$. Indeed, for $Y \in \mathcal{P}_{2n}$ and $A \in \widetilde{Sp}(2n;\mathbb{R})$ one has

$$\begin{aligned}(Y[A])^{-1}J = ({}^{\mathrm{t}}AYA)^{-1}J = A^{-1}Y^{-1}\,{}^{\mathrm{t}}A^{-1}J \sim A(A^{-1}Y^{-1}\,{}^{\mathrm{t}}A^{-1}J)A^{-1} \\ = Y^{-1}\,{}^{\mathrm{t}}A^{-1}JA^{-1} = \varepsilon(A)Y^{-1}J\end{aligned}$$

with $\varepsilon(A) \in \{\pm 1\}$. By the definition of the d_k we thus have $d_k(Y[A]) = d_k(Y)$ for all $1 \leqslant k \leqslant n$, $Y \in \mathcal{P}_{2n}$ and $A \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})$.

For the next proposition recall the following from Definition 1.17:

$$\mathcal{P}^*_{2n}(Y) = \left\{ Y[S] \mid S \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R}) \right\}$$

PROPOSITION 1.48. *Let $X, Y \in \mathcal{P}_{2n}$. Then*

$$X \in \mathcal{P}^*_{2n}(Y) \quad \textit{iff} \quad Y \in \mathcal{P}^*_{2n}(X) \quad \textit{iff} \quad d_j(X) = d_j(Y) \textit{ for all } 1 \leqslant j \leqslant n\,.$$

PROOF. The first 'iff' is a direct consequence of the definition of $\mathcal{P}^*_{2n}(M)$, $M \in \mathcal{P}_{2n}$.

We first prove the second 'if': By assumption $X^{-1}J \sim Y^{-1}J$. Since $X^{-1}J \sim X^{-1/2}JX^{-1/2}$, we have $X^{-1/2}JX^{-1/2} \sim Y^{-1/2}JY^{-1/2}$, which means that there is $A \in \mathrm{O}(2n;\mathbb{R})$ such that

$${}^{\mathrm{t}}AX^{-1/2}JX^{-1/2}A = Y^{-1/2}JY^{-1/2}\,.$$

This implies that $X^{-1/2}AY^{1/2} =: S \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})$. By definition of S we have $AY^{1/2} = X^{1/2}S$ and hence $Y = Y^{1/2}\,{}^{\mathrm{t}}AAY^{1/2} = \mathrm{Id}[AY^{1/2}] = \mathrm{Id}[X^{1/2}S] = {}^{\mathrm{t}}SXS$ as claimed. The 'only if' part is proved as follows: If $Y \in \mathcal{P}^*_{2n}(X)$ then $Y = {}^{\mathrm{t}}SXS$ for some $S \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})$. But then $Y^{-1}J = S^{-1}X^{-1}\,{}^{\mathrm{t}}S^{-1}J = \pm S^{-1}X^{-1}JS \sim \pm X^{-1}J$ since $S \in \widetilde{\mathrm{Sp}}(2n;\mathbb{R})$. $\square$

In Section 1 we defined data associated with a compact nilmanifold $(\Gamma\backslash G, \mathbf{m})$. In the following proposition we collect that data in case $(\Gamma\backslash G, \mathbf{m})$ is a normalised Heisenberg manifold. Recall the definition of the structure map (1.9) on page 11, the definition of the lattices $\Gamma_{\mathfrak{z}}$ and $\Gamma_{\mathfrak{n}}$ in (1.12) on page 15 and the definition of the fibre torus and the base torus in (1.13) on page 15. We also remind ourselves that every compact Heisenberg manifold is a Heisenberg-like nilmanifold.

PROPOSITION 1.49. *Let $n \in \mathbb{N}$, $r \in \mathcal{D}_n$ and $(\Gamma^r \backslash H_n, \mathbf{m})$ be a normalised Heisenberg manifold with $\mathbf{m} = (h, g)$. Then we have*

- $\Gamma^r_{\mathfrak{z}} = \mathbb{Z} \cdot Z$ *and* $T_{\mathfrak{z},\mathbf{m}_{\mathfrak{z}}} \simeq (\mathbb{Z}\backslash\mathbb{R}, g)$,
- $\Gamma^r_{\mathfrak{n}} = \delta_r \cdot \mathbb{Z}^{2n}$ *and* $T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}} = \left(\delta_r \cdot \mathbb{Z}^{2n}\backslash\mathbb{R}^{2n}, h\right) \simeq \left(\mathbb{Z}^{2n}\backslash\mathbb{R}^{2n}, h[\delta_r]\right)$,
- *relative to the basis* $\mathfrak{B}_n$, $j\left(g^{-1/2}Z\right)$ *has the representation* $-g^{1/2}h^{-1}J$,
- $c^{\mathbf{m}}_k = g^{1/2}d_k(h)$ *for all* $1 \leqslant k \leqslant n$.

PROOF. By the definition of Γ^r and (1.7) we have

$$\log \Gamma^r = \left\{ X(x, y, s - \tfrac{1}{2}\langle x, y\rangle) \mid \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \delta_r \cdot \mathbb{Z}^{2n}, s \in \mathbb{Z} \right\} .$$

By definition of $\Gamma^r_{\mathfrak{z}}$ and $\Gamma^r_{\mathfrak{n}}$ we have

$$\begin{aligned} \Gamma^r_{\mathfrak{z}} &= \mathfrak{z} \cap \log \Gamma^r = \{X(0,0,s) \mid s \in \mathbb{R}\} \cap \log \Gamma^r = \operatorname{span}_{\mathbb{Z}}\{Z\} , \\ \Gamma^r_{\mathfrak{n}} &= \pi_{\mathfrak{n}} \log \Gamma^r = \left\{ X(x,y,0) \mid \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \delta_r \cdot \mathbb{Z}^{2n} \right\} = \delta_r \cdot \mathbb{Z}^{2n}. \end{aligned}$$

By definition of the structure map j (see (1.9)) we have

$$\left\langle j\left(g^{-1/2}Z\right) X_i, Y_j \right\rangle_h = \left\langle g^{-1/2}Z, [X_i, Y_j] \right\rangle_g = g^{1/2}\delta_{i,j} ,$$

where $\delta_{i,j}$ is the Kronecker-delta symbol. Let A be the matrix representation of $j\left(g^{-1/2}Z\right)$ w.r.t. the basis $\mathfrak{B}_n$. Since brackets of the form $[X_i, X_j]$ and $[Y_i, Y_j]$ vanish, we have ${}^{\mathrm{t}}Ah = g^{1/2}J$ from which we obtain

$$A = -g^{1/2}h^{-1}J .$$

The expression for the $c^{\mathbf{m}}_k$ now follow from Definition 1.47*(i)*. □

COROLLARY 1.50. *The normalised Heisenberg manifold $(\Gamma^r\backslash H_n, \mathbf{m})$ with $\mathbf{m} = (h, g)$ is of Heisenberg type if and only if $h \in \mathcal{P}^*_{2n}\left(g^{1/2}\mathrm{Id}\right)$ if and only if $d_k(h) = g^{-1/2}$ for all $1 \leqslant k \leqslant n$. Accordingly, the set*

$$\mathcal{M}^{HT}_n := \bigcup_{r \in \mathcal{D}_n} \mathcal{M}^{r,HT}_n , \quad \mathcal{M}^{r,HT}_n := \left\{ [(h,g)] \in \mathcal{M}^r_n \mid h \in \mathcal{P}^*_{2n}(g^{1/2}\mathrm{Id}) \right\}$$

parametrises the isometry classes of compact Nilmanifolds of Heisenberg type with one dimensional centre. Furthermore, the restriction of the homeomorphism $\mathcal{M}^r_n \to \mathscr{M}(\Gamma^r, H_n)$ from Theorem 1.46 to $\mathcal{M}^{r,HT}_n$ is a homeomorphism onto its image, denoted by $\mathscr{M}^{HT}(\Gamma^r, H_n)$.

PROOF. By the last proposition $(\Gamma^r\backslash H_n, \mathbf{m})$ is of Heisenberg type if and only if $d_k(h) = g^{-1/2}$ for all $1 \leqslant k \leqslant n$. This, in turn, is the case if and only if $h \in \mathcal{P}^*_{2n}\left(g^{1/2}\mathrm{Id}\right)$ by Proposition 1.48.

That the $\mathcal{M}^{r,HT}_n$ parametrise the compact nilmanifolds of Heisenberg type with one dimensional centre now follows from Theorem 1.46. □

We will now study the compact sets of the moduli space $\mathcal{M}_n^r$.

NOTATION AND REMARKS 1.51.

(i) *Minkowski's fundamental domain* in $\mathcal{P}_n$ is the domain

$$\mathscr{M}_n = \{Y = (y_{i,j}) \in \mathcal{P}_n \mid \forall k = 1, \dots, n : y_{k,k+1} \geqslant 0 \\ \text{and } Y[a] \geqslant y_{k,k} \text{ for all } a \in \mathbb{Z}^n \text{ with } \gcd(a_k, \dots, a_n) = 1\}.$$

(ii) For $Y \in \mathcal{P}_n$ we define

$$m(Y) := \inf\{Y[a] \mid a \in \mathbb{Z}^n \backslash \{0\}\}.$$

The value $m(Y)$ is called *the first minimum of* Y. It is the squared norm of a shortest nonzero vector of a lattice with Gram matrix Y. Note that $m(Y) = y_{1,1}$ for $Y \in \mathscr{M}_n$ by the very definition of $\mathscr{M}_n$.

(iii) For $r \in \mathcal{D}_n$ and $Y \in \mathcal{P}_{2n}$ we define

$$m_r(Y) := \inf\{Y[\delta_r a] \mid a \in \mathbb{Z}^{2n} \backslash \{0\}\} = m(Y[\delta_r]).$$

If $r = (1, \dots, 1)$, we abbreviate $m_r(Y)$ simply to $m(Y)$ which is in accordance with (ii).

(iv) The function $m : \mathcal{P}_n \to (0, \infty)$ is constant on the orbits of the action of $\mathrm{GL}(n; \mathbb{Z})$ on $\mathcal{P}_n$ and we denote the induced function on $\mathcal{P}_n / \mathrm{GL}(n; \mathbb{Z})$ by m, too.

Similarly, the function $m_r : \mathcal{P}_{2n} \to (0, \infty)$ is constant on the orbits of the action of $G_r = \delta_r \mathrm{GL}(2n; \mathbb{Z}) \delta_r^{-1}$ on $\mathcal{P}_{2n}$ and we denote the induced function on the quotient by m_r as well.

THEOREM 1.52 ([**Ter88**, 4.4.2 Theorem 1]). *Minkowski's fundamental domain $\mathscr{M}_n$ has the following properties:*

(a) For any $Y \in \mathcal{P}_n$, there exists a matrix $A \in \mathrm{GL}(n; \mathbb{Z})$ such that $Y[A]$ lies in $\mathscr{M}_n$.

(b) Only a finite number of inequalities are neccessary in the definition of $\mathscr{M}_n$. Thus, $\mathscr{M}_n$ is a convex cone through the origin bounded by a finite number of hyperplanes.

(c) If Y and $Y[A]$ both lie in the domain $\mathscr{M}_n$, and A is an element of $\mathrm{GL}(n; \mathbb{Z})$ distinct from $\pm\mathrm{Id}$, then Y must lie on the boundary $\partial\mathscr{M}_n$. Moreover, $\mathscr{M}_n$ is bounded by a finite number of images $\mathscr{M}_n[A]$, for A in $\mathrm{GL}(n; \mathbb{Z})$. That is $\mathscr{M}_n \cap (\mathscr{M}_n[A]) \neq \varnothing$ for only finitely many $A \in \mathrm{GL}(n; \mathbb{Z})$.

REMARK 1.53. Theorem 1.52 states that $\mathscr{M}_n$ is a fundamental domain for the quotient space $\mathcal{P}_n / \mathrm{GL}(n; \mathbb{Z})$. For $r \in \mathcal{D}_n$, define the map

$$\Psi_r : \mathcal{P}_{2n} \to \mathcal{P}_{2n} \tag{1.30}$$

$$Y \mapsto Y\left[\delta_r^{-1}\right] = \delta_r^{-1} Y \delta_r^{-1}\,.$$

Then Ψ_r induces a map

$$\begin{aligned}\psi_r : \mathcal{P}_{2n}/\operatorname{GL}(2n;\mathbb{Z}) &\to \mathcal{P}_{2n}/G_r \\ [Y] &\mapsto [Y[\delta_r^{-1}]] = [\delta_r^{-1} Y \delta_r^{-1}]\,.\end{aligned} \tag{1.31}$$

The maps Ψ_r and ψ_r are diffeomorphisms and satisfy $\pi_r \circ \Psi_r = \psi_r \circ \pi$ where $\pi : \mathcal{P}_{2n} \to \mathcal{P}_{2n}/\operatorname{GL}(2n;\mathbb{Z})$ and $\pi_r : \mathcal{P}_{2n} \to \mathcal{P}_{2n}/G_r$ are the canonical projections. It follows that

$$\mathscr{M}_{2n,r} := \Psi_r(\mathscr{M}_{2n}) \tag{1.32}$$

is a fundamental domain for the space $\mathcal{P}_{2n}/G_r$. Note that $m_r(\Psi_r(Y)) = m(Y)$ and $m_r(\psi_r([Y])) = m([Y])$.

PROPOSITION 1.54 ([**Ter88**, 4.4.2 Proposition 1]).
(a) If $Y \in \mathscr{M}_n$, then the entries of Y satisfy the inequalities

$$y_{1,1} \leqslant y_{2,2} \leqslant \cdots \leqslant y_{n,n}$$

and

$$|y_{i,j}| \leqslant y_{i,i}/2 \quad \text{if} \quad 1 \leqslant i < j \leqslant n\,.$$

(b) There is a positive constant k_n such that all $Y \in \mathscr{M}_n$ satisfy the inequality

$$k_n y_{1,1} \cdots y_{n,n} \leqslant \det Y \leqslant y_{1,1} \cdots y_{n,n}\,.$$

The right hand inequality actually holds for any $Y \in \mathcal{P}_n$.

REMARK 1.55. We will in later sections often make an argument of the following form: If $\det Y \to 0$ for a sqeuence or a path $Y \in \mathcal{P}_n$, then $m(Y) \to 0$. We justify this argument as follows: without loss of generality we can assume that $Y \in \mathscr{M}_n$ ($Y[G] \in \mathscr{M}_n$ for some $G \in \operatorname{GL}(n;\mathbb{Z})$, under whose action $\det(\cdot)$ and $m(\cdot)$ are invariant). By part *(b)* of the last proposition, $\det Y \to 0$ implies that $y_{1,1} \cdots y_{n,n} \to 0$, and since, by part *(a)*, $y_{1,1} \leqslant \ldots \leqslant y_{n,n}$, surely $y_{1,1} \to 0$. Finally, by Remark 1.51(ii), we have $y_{1,1} = m(Y)$.

THEOREM 1.56 (Mahler-Hermite Compactness Theorem). *Any set $M \subset \mathscr{M}_n$ for which there are positive constants $C_0, C_1 > 0$ such that $m(Y) \geqslant C_0$ and $\det Y \leqslant C_1$ for all $Y \in M$ has compact closure in $\mathscr{M}_n$.*

PROOF. By assumption $\det Y$ is bounded from above by C_1 and $m(Y)$ is bounded from below by C_0 on M. It follows from Remarks 1.51 (ii) and Proposition 1.54 that the diagonal entries of Y are bounded from below by C_0 and from above by $k_n^{-1} C_0^{1-n} C_1$. This in turn, again by Proposition 1.54, implies that all entries of Y are bounded from below and above. Hence, the closure $\overline{M}$ is compact in $\mathrm{Sym}(n;\mathbb{R})$, the space of symmetric matrices. But, since $0 < k_n C_0^n \leqslant k_n m(Y)^n \leqslant det(Y)$ for all $Y \in \overline{M}$, we already have $\overline{M} \subset \mathcal{P}_n$. Finally, by Theorem 1.52*(c)*, we know that Minkowski's fundamental domain $\mathscr{M}_n$ is a closed cone with boundary the union of a finite number of hypersurfaces. Hence, $\overline{M} \subset \mathscr{M}_n$. □

COROLLARY 1.57 (Selection Theorem of Mahler, cf. [**GL87**, Ch. 3 §17 Theorem 2]).
Let $M \subset \mathcal{P}_n / \mathrm{GL}(n;\mathbb{Z})$ be a set such that there are constants $C_0, C_1 > 0$ so that $m([Y]) \geqslant C_0$ and $\det([Y]) \leqslant C_1$ for all $[Y] \in M$. Then M has compact closure in $\mathcal{P}_n / \mathrm{GL}(n;\mathbb{Z})$.

COROLLARY 1.58. *Let $n \in \mathbb{N}$, $r \in \mathcal{D}_n$ and $M \subset \mathcal{P}_{2n}/G_r$ be a set such that there are constants $C_0, C_1 > 0$ so that $m_r([Y]) \geqslant C_0$ and $\det([Y]) \leqslant C_1$ for all $[Y] \in M$. Then M has compact closure in $\mathcal{P}_{2n}/G_r$.*

PROOF. This follows from the last corollary via the diffeomorphism ψ_r defined in Remark 1.53. □

The following example shows that Corollary 1.58 does not remain true if we replace $\mathcal{P}_{2n}/G_r$ by $\mathcal{P}_{2n}/\Pi_r = \mathcal{P}_{2n}/(G_r \cap \widetilde{\mathrm{Sp}}(2n;\mathbb{R}))$.

EXAMPLE 1.59. Let $k \in \mathbb{N}_0$ and

$$Y_k := \begin{pmatrix} 1 & k & 0 & 0 \\ k & k^2+1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = {}^{\mathrm{t}}\!\begin{pmatrix} 1 & k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \mathrm{Id} \cdot \begin{pmatrix} 1 & k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Obviously, $\det(Y_k) = 1$ for all $k \in \mathbb{N}_0$. Also, since Y_k is in the same $\mathrm{GL}(4;\mathbb{Z})$-orbit as Id, $m(Y_k) = 1$ for all $k \in \mathbb{N}_0$. One easily calculates

$$Y_k^{-1} \cdot J = \begin{pmatrix} k^2+1 & -k & 0 & 0 \\ -k & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot J = \begin{pmatrix} 0 & 0 & k^2+1 & -k \\ 0 & 0 & -k & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

so that

$$\left(Y_k^{-1} \cdot J\right)^2 = \begin{pmatrix} -k^2-1 & k & 0 & 0 \\ k & -1 & 0 & 0 \\ 0 & 0 & -k^2-1 & k \\ 0 & 0 & k & -1 \end{pmatrix}.$$

The eigenvalues of $\left(Y_k^{-1}\cdot J\right)^2$ are thus the solutions of

$$0=\left((X+k^2+1)(X+1)-k^2\right)^2=\left(X^2+(k^2+2)X+1\right)^2 .$$

It follows that

$$d_1(Y_k)=2^{-1/2}\sqrt{k^2+2-k\sqrt{k^2+4}}\,,$$
$$d_2(Y_k)=2^{-1/2}\sqrt{k^2+2+k\sqrt{k^2+4}}\,.$$

The sequence $d_2(Y_k)$ is monotonously and unboundedly increasing in k. Since d_2 is an invariant of the $\widetilde{\mathrm{Sp}}(4;\mathbb{R})$-action (see Remarks 1.47(iii)), no two matrices of the family $\{Y_k\}_{k\in\mathbb{N}}$ are in the same $\widetilde{\mathrm{Sp}}(4;\mathbb{Z})$-orbit. Note that the $d_j(Y)$, $j=1,2$, are continuous in $Y\in\mathcal{P}_4$ (see Remarks 1.47(ii)) and descend to continuous functions on $\mathcal{P}_4/\widetilde{\mathrm{Sp}}(4;\mathbb{Z})$. Therefore, no subsequence of $\{[Y_k]\}_{k\in\mathbb{N}}$ converges. In particular, boundedness of $\det(\cdot)$ and $m_{(1,1)}(\cdot)$ on a set $M\subset\mathcal{P}_4/\Pi_{(1,1)}$ is not sufficient for M to be relatively compact.

LEMMA 1.60. *Let $n\in\mathbb{N}$, $r\in\mathcal{D}_n$ and let U be a complete set of representatives for G_r/Π_r. Then, for any $C>0$ and any matrix norm $\|\cdot\|:M(2n;\mathbb{R})\to[0,\infty)$ there are only finitely many $G\in U$ with $\|{}^{\mathrm{t}}G^{-1}JG^{-1}\|\leqslant C$.*

PROOF. Firstly, for any $G,H\in U$ with $G\neq H$ we have ${}^{\mathrm{t}}G^{-1}JG^{-1}\neq\pm{}^{\mathrm{t}}H^{-1}JH^{-1}$. For if this were not the case, we would have $H^{-1}G=P\in\widetilde{\mathrm{Sp}}(2n;\mathbb{R})$, that is $[G]=[HP]=[H]\in G_r/\Pi_r$ which would be a contradiction. Secondly, if $G\in U$, then the entries of ${}^{\mathrm{t}}G^{-1}JG^{-1}$ are elements of $\frac{1}{r_n^2}\mathbb{Z}$. Since $M(2n;\mathbb{R})$ is a finite dimensional vector space, all norms are equivalent and we can choose a particular one. Let $\|\cdot\|$ be the maximum norm $\|G\|=\max\{|G_{i,j}|\,|\,1\leqslant i,j\leqslant 2n\}$. Then $\|{}^{\mathrm{t}}G^{-1}JG^{-1}\|\geqslant r_n^{-2}$ and by the above, $\|{}^{\mathrm{t}}G^{-1}JG^{-1}-{}^{\mathrm{t}}H^{-1}JH^{-1}\|\geqslant r_n^{-2}$ for all $G,H\in U$ with $G\neq H$. The lemma's statement now follows from the fact that the closed norm ball $\{G\in M(2n;R)\,|\,\|G\|\leqslant C\}$ is compact. ☐

NOTATION AND REMARKS 1.61.

(i) By the spectral theorem for symmetric matrices the eigenvalues of any matrix $Y\in\mathcal{P}_n$ are real and positive. We introduce functions $\lambda_1,\dots,\lambda_n:\mathcal{P}_n\to(0,\infty)$ such that $\lambda_k(Y)$ is an eigenvalue of Y and $\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_n$. Note that the λ_k, $1\leqslant k\leqslant n$, are continuous functions (see, e.g., [**Zed65**, Theorem 1]).

(ii) We furthermore define functions $s_1,\dots,s_n:\mathrm{GL}(n;\mathbb{R})\to(0,\infty)$ such that $s_k(G)$, $1\leqslant k\leqslant n$, are the *singular values* of G and $s_1\geqslant s_2\geqslant\cdots\geqslant s_n$. These functions satisfy $s_k(G)=\sqrt{\lambda_{n-k+1}({}^{\mathrm{t}}GG)}$ which means in particular that they are continuous. Care has to be taken though as the s_k are in general not invariant under conjugation.

If, however, ${}^{\mathrm{t}}GG \sim A \in \mathcal{P}_n$ then $s_k(G)$, $1 \leqslant k \leqslant n$, are the square roots of the eigenvalues of A.

(iii) We want to relate the functions $d_j : \mathcal{P}_{2n}$, $1 \leqslant j \leqslant n$, from Definition 1.47 to the singular values $s_k : \mathrm{GL}(2n;\mathbb{R}) \to (0,\infty)$, $1 \leqslant k \leqslant 2n$. For any $G \in \mathrm{GL}(2n;\mathbb{R})$ we have

$$\begin{aligned} {}^{\mathrm{t}}\left({}^{\mathrm{t}}G^{-1}JG^{-1}\right)\left({}^{\mathrm{t}}G^{-1}JG^{-1}\right) &= -\,{}^{\mathrm{t}}G^{-1}JG^{-1}\,{}^{\mathrm{t}}G^{-1}JG^{-1} \\ &\sim -G^{-1}\,{}^{\mathrm{t}}G^{-1}JG^{-1}\,{}^{\mathrm{t}}G^{-1}J = -\left(\left({}^{\mathrm{t}}GG\right)^{-1}J\right)^2, \end{aligned}$$

which implies

$$d_j({}^{\mathrm{t}}GG) = s_k({}^{\mathrm{t}}G^{-1}JG^{-1})$$

for $k \in \{2n-2j+2, 2n-2j+1\}$.

LEMMA 1.62. *The inequality*

$$d_n\left({}^{\mathrm{t}}GG\right)\left(\lambda_{2n}(Y)\right)^{-1} \leqslant d_n(Y[G])$$

holds for all $Y \in \mathcal{P}_{2n}$ *and* $G \in \mathrm{GL}(2n;\mathbb{R})$.

PROOF. By [**Bha96**, p. 72, (III.20)] one has

$$\prod_{j=1}^{k} s_{i_j}(A) \prod_{j=1}^{k} s_{2n-i_j+1}(B) \leqslant \prod_{j=1}^{k} s_j(AB) \tag{1.33}$$

for all $A, B \in M(2n;\mathbb{R})$ and all $1 \leqslant i_1 < \cdots < i_k \leqslant 2n$. We choose $k = 1$, $i_1 = 2n$, $A = Y^{-1/2}$ and $B = {}^{\mathrm{t}}G^{-1}JG^{-1}Y^{-1/2}$ and obtain

$$s_{2n}\left(Y^{-1/2}\right) s_1\left({}^{\mathrm{t}}G^{-1}JG^{-1}Y^{-1/2}\right) \leqslant s_1\left(Y^{-1/2}\,{}^{\mathrm{t}}G^{-1}JG^{-1}Y^{-1/2}\right).$$

We apply (1.33) again to the second factor of the left hand side of this inequality, this time with $A = {}^{\mathrm{t}}G^{-1}JG^{-1}$, $B = Y^{-1/2}$, $k = 1$ and $i_1 = 1$, which yields

$$\left(s_{2n}\left(Y^{-1/2}\right)\right)^2 s_1\left({}^{\mathrm{t}}G^{-1}JG^{-1}\right) \leqslant s_1\left(Y^{-1/2}\,{}^{\mathrm{t}}G^{-1}JG^{-1}Y^{-1/2}\right). \tag{1.34}$$

Now $\left(s_{2n}\left(Y^{-1/2}\right)\right)^2 = s_{2n}\left(Y^{-1}\right) = \lambda_1\left(Y^{-1}\right) = (\lambda_{2n}(Y))^{-1}$. Furthermore, one has $s_1\left({}^{\mathrm{t}}G^{-1}JG^{-1}\right) = d_n({}^{\mathrm{t}}GG)$ by Remark 1.61(iii). Together, this shows that the left hand side of (1.34) matches the left hand side of the inequality in the statement of the lemma. We have a look at the right hand side of (1.34). Let $H := Y^{1/2}G$. By Remark 1.61(iii) we have

$$\begin{aligned} s_1\left(Y^{-1/2}\,{}^{\mathrm{t}}G^{-1}JG^{-1}Y^{-1/2}\right) = s_1\left({}^{\mathrm{t}}H^{-1}JH^{-1}\right) = d_n\left({}^{\mathrm{t}}HH\right) = d_n\left({}^{\mathrm{t}}GY^{1/2}Y^{1/2}G\right) \\ = d_n\left(Y[G]\right), \end{aligned}$$

which finishes the proof of the stated inequality. □

THEOREM 1.63. *Let $M \subset \mathcal{P}_{2n}/\Pi_r$. Assume that there are positive constants C_0, C_1 and C_2 such that $m_r([Y]) \geqslant C_0$, $\det([Y]) \leqslant C_1$ and $d_n([Y]) \leqslant C_2$ for all $[Y] \in M$. Then, M has compact closure in $\mathcal{P}_{2n}/\Pi_r$.*

PROOF. We denote by $\pi_r : \mathcal{P}_{2n} \to \mathcal{P}_{2n}/G_r$, $p_r : \mathcal{P}_{2n} \to \mathcal{P}_{2n}/\Pi_r$ and $\eta_r : \mathcal{P}_{2n}/\Pi_r \to \mathcal{P}_{2n}/G_r$ the canonical projections. Note that $\eta_r \circ p_r = \pi_r$.
By Corollary 1.58, the set $\eta_r(M)$ is precompact. There is thus a precompact set $K \subset \mathscr{M}_{2n,r}$, where $\mathscr{M}_{2n,r}$ is the fundamental domain for $\mathcal{P}_{2n}/\Pi_r$ defined by (1.32), with $\pi_r(K) = \eta_r(M)$. Consequently, one has

$$M \subseteq \bigcup_{G \in U} p_r\left({}^{\mathrm{t}}GKG\right) , \tag{1.35}$$

where U is a full set of representatives of G_r/Π_r. The function λ_{2n} is continuous and therefore takes its maximum $\overline{M} > 0$ on the closure $\overline{K}$ of K. By Lemma 1.62 one has

$$d_n\left({}^{\mathrm{t}}GYG\right) \geqslant (\lambda_{2n}(Y))^{-1} d_n\left({}^{\mathrm{t}}GG\right) \geqslant \overline{M}^{-1} d_n\left({}^{\mathrm{t}}GG\right) \text{ for all } Y \in K, G \in U . \tag{1.36}$$

Let $\|\cdot\|_2$ be the spectral norm, that is, $\|G\|_2 = s_1(G)$. Also, recall that $d_n\left({}^{\mathrm{t}}GG\right) = s_1\left({}^{\mathrm{t}}G^{-1}JG^{-1}\right)$ by Remark 1.61(iii). Then, by Lemma 1.60 there are only finitely many $G \in U$ with $d_n\left({}^{\mathrm{t}}GG\right) = s_1\left({}^{\mathrm{t}}G^{-1}JG^{-1}\right) = \|{}^{\mathrm{t}}G^{-1}JG^{-1}\|_2 \leqslant C_2\overline{M}$. Let $G_1, \ldots, G_N$ be those $G \in U$. Because of inequality (1.36) we have

$$d_n\left({}^{\mathrm{t}}GYG\right) > C_2 \text{ for all } Y \in K \text{ and all } G \in U \backslash \{G_1, \ldots, G_N\} ,$$

which in turn implies, by the assumption on $d_{n|M}$, that

$$M \subseteq \bigcup_{j=1}^{N} p_r\left({}^{\mathrm{t}}G_jKG_j\right) .$$

The right hand side is a finite union of precompact sets and hence precompact. The set M has thus compact closure, as claimed. □

For the next corollary, recall the imbedding of $\widetilde{\mathrm{Sp}}(2n; \mathbb{R})$ into $\mathrm{GL}(2n+1; \mathbb{R})$ by (1.23) and the moduli space $\mathcal{M}_n^r$ defined in Theorem 1.46.

COROLLARY 1.64. *Let $n \in \mathbb{N}$, $r \in \mathcal{D}_n$ and $M \subset \mathcal{M}_n^r$. Assume that there are positive constants $C_0, C_1, C_2 > 0$ and a compact intervall $I \subset (0, \infty)$ such that $g \in I$, $m_r(h) \geqslant C_0$, $\det(h) \leqslant C_1$ and $d_n(h) \leqslant C_2$ for all $[(h, g)] \in M$. Then M has compact closure.*

COROLLARY 1.65. *Let $n \in \mathbb{N}$ and $r \in \mathcal{D}_n$. Then any set $M \subset \mathcal{M}_n^{r,HT}$ for which there exists a constant $C_0 > 0$ and a compact interval $I \subset (0, \infty)$ such that $m_r([h]) \geqslant C_0$ and $g \in I$ for all $[(h, g)] \in M$ is relatively compact.*

PROOF. By Proposition 1.40 and Theorem 1.46, $\mathcal{M}_n^{r,HT}$ is a closed subspace of $\mathcal{M}_n^r$. Hence, the closure of M is still contained in $\mathcal{M}_n^{r,HT}$. □

REMARK 1.66. We think of the degeneracies $m_r(h) = 0$, $det(h) = \infty$, $d_n(h) = \infty$ and $g \in \{0, \infty\}$ as defining the boundary (at infinity) $\partial_\infty \mathcal{M}_n^r = \partial \mathcal{M}_n^r$ of $\mathcal{M}_n^r$. In later sections we will speak of the behaviour of a function as the argument $\mathbf{m} = (h, g) \in \mathcal{M}_n^r$ (or one of the subspaces of $\mathcal{M}_n^r$) approaches the boundary, by which we mean that $\mathbf{m}$ degenerates in one of these ways.

J. M. Lee **[Lee02]** introduces the notion of a sequence *escaping to infinity*: The sequence $\{x_i\}_{i\in\mathbb{N}} \subset T$, where T is a topological manifold, is said to escape to infinity if for every compact set $K \subset T$, there are only finitely many x_i containted in K. By our definition above and Corollary 1.64, the sequences escaping to infinity in $\mathcal{M}_n^r$ are precisely the sequences approaching the boundary at infinity.

We state the result of Corollary 1.64 for an arbitrary nilmanifold $\Gamma\backslash G$, where G has one-dimensional centre, by pulling the geometry to $\Gamma^r\backslash H_n$ via the homeomorphism defined in Theorem 1.46.

DEFINITION 1.67. Let $(V, \langle\cdot,\cdot\rangle)$ be a euclidean vector space and $L \subset V$ a lattice of full rank. We define

$$m(L) := m(L, \langle\cdot,\cdot\rangle) := \inf\left\{\|v\|^2 \mid v \in L\backslash\{0\}\right\}.$$

Note that this is in accordance with Notation 1.51(ii) in the sense that if $Y \in \mathcal{P}_{\dim V}$ is a Gram matrix for L, then $m(L) = m(Y)$.

COROLLARY 1.68. *Let G be a connected and simply connected 2-step nilpotent Lie group of dimension $2n+1$ with Lie algebra $\mathfrak{g}$, $\Gamma \subset G$ a uniform subgroup and $M \subset \mathscr{M}^{HL}(\Gamma, G)$. Assume $\dim \mathfrak{z} = 1$ and that there are constants $C_i > 0$, $i = 0, \ldots, 4$, such that $m(\Gamma_\mathfrak{z}, \mathbf{m}_\mathfrak{z}) > C_0$, $\operatorname{Vol} T_{\mathfrak{z},\mathbf{m}_\mathfrak{z}} < C_1$, $m(\Gamma_\mathfrak{n}, \mathbf{m}_\mathfrak{n}) > C_2$, $\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}} < C_3$ and $c_n^{\mathbf{m}} < C_4$ for all $[\mathbf{m}] \in M$. Then M has compact closure in $\mathscr{M}^{HL}(\Gamma, G)$.*

PROOF. By assumption, the centre of G is one dimensional. Hence, w.l.o.g., $G = H_n$, the $(2n+1)$-dimensional Heisenberg group. By Theorem 1.44, we can further assume that $\Gamma = \Gamma^r$ for some $r \in \mathcal{D}_n$. We now identify the set M with its preimage in $\mathcal{M}_n^r$ under the homeomorphism from Theorem 1.46. Let $[\mathbf{m}] \in M$ with $\mathbf{m} = (h, g)$. By Proposition 1.49 and Remark 1.53 we have

$$m(\Gamma_\mathfrak{z}, \mathbf{m}_\mathfrak{z}) = m(\mathbb{Z}, g) = g\,, \operatorname{Vol} T_{\mathfrak{z},\mathbf{m}_\mathfrak{z}} = \operatorname{Vol}(\mathbb{Z}\backslash\mathbb{R}, g) = \sqrt{g}\,,$$

$$m(\Gamma_\mathfrak{n}, \mathbf{m}_\mathfrak{n}) = m(\delta_r \cdot \mathbb{Z}^{2n}, h) = m_r(h)\,, \operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}} = \operatorname{Vol}(\delta_r \cdot \mathbb{Z}^{2n}\backslash\mathbb{R}^{2n}, h) = \left(\prod_{k=1}^n r_k\right)\sqrt{\det h}\,,$$

$$c_n^{\mathbf{m}} = g^{1/2} d_n(h)\,.$$

The statement of the corollary now follows from Corollary 1.64. □

REMARK 1.69.

(i) It seems reasonable to conjecture that the conclusion of Corollary 1.68 still holds if one drops the assumption that the centre of G is one dimensional.

(ii) In accordance with *(i)*, for an arbitrary connected and simply connected nonsingular 2-step nilpotent Lie group G and a uniform subgroup Γ, we define the boundary (at infinity) $\partial_\infty \mathscr{M}^{HL}(\Gamma, G) = \partial \mathscr{M}^{HL}(\Gamma, G)$ of $\mathscr{M}^{HL}(\Gamma, G)$ (resp. its subspaces) by the degeneracies $m(\Gamma_{\mathfrak{z}}, \mathbf{m}_{\mathfrak{z}}) = 0$, $\mathrm{Vol}\, T_{\mathfrak{z},\mathbf{m}_{\mathfrak{z}}} = \infty$, $m(\Gamma_{\mathfrak{n}}, \mathbf{m}_{\mathfrak{n}}) = 0$, $\mathrm{Vol}\, T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}} = \infty$ and $c_n^{\mathbf{m}} = \infty$.

DEFINITION 1.70. Let G be a connected and simply connected nonsingular 2-step nilpotent Lie group and $\Gamma \subset G$ a uniform subgroup. We define subsets of the moduli spaces $\mathscr{M}(\Gamma, G)$, $\mathscr{M}^{HL}(\Gamma, G)$ and $\mathscr{M}^{HT}(\Gamma, G)$ that will come into play in Chapter 2. Let

$$\begin{aligned}
\mathscr{S}\mathscr{M}(\Gamma, G) &:= \{[\mathbf{m}] \in \mathscr{M}(\Gamma, G) \mid \mathrm{Vol}(\Gamma\backslash G, \mathbf{m}) = 1\}\,,\\
\mathscr{S}\mathscr{M}^{HL}(\Gamma, G) &:= \mathscr{S}\mathscr{M}(\Gamma, G) \cap \mathscr{M}^{HL}(\Gamma, G)\,,\\
\mathscr{S}\mathscr{M}^{HT}(\Gamma, G) &:= \mathscr{S}\mathscr{M}(\Gamma, G) \cap \mathscr{M}^{HT}(\Gamma, G)\,.
\end{aligned}$$

For any $C > 0$ we also define

$$\mathscr{S}\mathscr{M}^{HL}_C(\Gamma, G) := \left\{[\mathbf{m}] \in \mathscr{S}\mathscr{M}^{HL}(\Gamma, G) \mid K_\sigma^{\mathbf{m}} \leqslant C\right\}\,,$$

where $K_\sigma^{\mathbf{m}}$ is the sectional curvature of $(\Gamma\backslash G, \mathbf{m})$. The sectional curvature is continuous in the metric. Hence, $\mathscr{S}\mathscr{M}^{HL}_C(\Gamma, G)$ is a closed subspace of $\mathscr{S}\mathscr{M}^{HL}(\Gamma, G)$.

In case $G = H_n$ and $\Gamma = \Gamma^r$ we also define subspaces of $\mathcal{M}_n^r$ that correspond to above defined moduli spaces under the homeomorphism from Theorem 1.46:

$$\begin{aligned}
\mathcal{S}\mathcal{M}_n^r &:= \{[(h,g)] \in \mathcal{M}_n^r \mid \mathrm{Vol}(\Gamma^r\backslash H_n, (h,g)) = 1\}\,,\\
\mathcal{S}\mathcal{M}_n^{r,HT} &:= \mathcal{M}_n^{r,HT} \cap \mathcal{S}\mathcal{M}_n^r = \left\{[(h,g)] \in \mathcal{S}\mathcal{M}_n^r \mid h \in \mathcal{P}^*_{2n}(g^{1/2}\mathrm{Id})\right\}\\
\mathcal{S}\mathcal{M}_{n,C}^r &:= \left\{[(h,g)] \in \mathcal{S}\mathcal{M}_n^r \mid K_\sigma^{(h,g)} \leqslant C\right\}\,,
\end{aligned}$$

where $\mathcal{M}_n^{r,HT}$ is as in Corollary 1.50.

Recall that $\mathscr{M}(\Gamma^r, H_n) = \mathscr{M}^{HL}(\Gamma^r, H_n)$. Under the homeomorphism from Theorem 1.46 we have $\mathcal{S}\mathcal{M}_n^r \simeq \mathscr{S}\mathscr{M}^{HL}(\Gamma^r, H_n)$, $\mathcal{S}\mathcal{M}_n^{r,HT} \simeq \mathscr{S}\mathscr{M}^{HT}(\Gamma^r, H_n)$ and $\mathcal{S}\mathcal{M}_{n,C}^r \simeq \mathscr{S}\mathscr{M}^{HL}_C(\Gamma^r, H_n)$.

REMARK 1.71. Let $n \in N$, $r \in \mathcal{D}_n$ and let $(\Gamma^r \backslash H_n, (h, g))$ be a normalised Heisenberg manifold. By Corollary 1.11 and Proposition 1.49 we have

$$\mathrm{Vol}(\Gamma^r \backslash H_n, (h, g)) = |\Gamma^r| \cdot \sqrt{g \cdot \det h}\,,$$

where $|\Gamma^r| := r_1 \cdot r_2 \cdots r_n$. Furthermore, by Definition 1.47(i) we have

$$\det(h) = \det(J \cdot h) = d_1(h)^{-2} \cdots d_n(h)^{-2}\,.$$

Now assume $(\Gamma^r \backslash H_n, (h, g))$ to have volume one and to be of Heisenberg type. By Corollary 1.50, the latter is the case if and only if $d_1(h) = \ldots = d_n(h) = g^{-1/2}$. Hence, $1 = \mathrm{Vol}(\Gamma^r \backslash H_n, (h, g)) = |\Gamma^r| \cdot g^{(n+1)/2}$. It follows that

$$\mathcal{SM}_n^{r,HT} = \left\{ [(h, g)] \in \mathcal{M}_n^r \mid g = |\Gamma^r|^{-2/(n+1)} \text{ and } h \in \mathcal{P}_{2n}^*(g^{1/2} \cdot \mathrm{Id}) \right\}\,.$$

In particular, $\mathcal{SM}_n^{r,HT} \ni [(h, g)] \mapsto \det h$ is constant.

6. A Poisson Type Formula

In this section we prove a Poisson type formula for all compact nonsingular Heisenberg-like nilmanifolds (Theorem 1.78). This formula will play a central role when we meromorphically continue the spectral ζ-function of nilmanifolds in the next section.

Let $T = L \backslash \mathbb{R}^n$ be a flat torus and $f : \mathbb{R}^n \ni x \mapsto e^{-4\pi^2 \|x\|^2 t} \in \mathbb{R}$ with $t > 0$. Then the classic Poisson summation formula from Corollary 1.35 applied to L^* and f reads, by 1.29*(vii)* and 1.30

$$\sum_{\lambda \in L^*} e^{-4\pi^2 \|\lambda\|^2 t} = \frac{\mathrm{Vol}\, T}{(4\pi t)^{n/2}} \sum_{X \in L} e^{-\frac{\|X\|^2}{4t}}\,. \tag{1.37}$$

Since $4\pi^2 \|\lambda\|^2$ with $\lambda \in L^*$ are the eigenvalues of the Laplace operator on T, the left hand side is precisely the heat trace $K(t)$ of T. Formula (1.37) has many applications. For example, $\{\|X\| \mid X \in L\}$ is the set of length of smoothly closed geodesics and with (1.37) one can retrieve each $\|X\|$, including its multiplicity, once one knows the left hand side, i.e., the Laplace spectrum of T. We are interested in (1.37) since it allows us to determine the behaviour of $K(t)$ for $t \searrow 0$. We write the right hand side of above formula as

$$\frac{\mathrm{Vol}\, T}{(4\pi t)^{n/2}} + \frac{\mathrm{Vol}\, T}{(4\pi t)^{n/2}} \sum_{X \in L \backslash \{0\}} e^{-\frac{\|X\|^2}{4t}}\,.$$

Then one can show that the second term is in $o(t^N)$ as $t \searrow 0$ for every $N \in \mathbb{N}$. Hence, the first term determines the behaviour of $K(t)$ for small t.

In Theorem 1.78 we will prove a Poisson type formula for every compact nonsingular Heisenberg-like nilmanifold $(\Gamma\backslash G, \mathbf{m})$:

$$K(t) = \frac{\mathrm{Vol}(\Gamma\backslash G)}{(4\pi t)^{\dim G/2}} \sum_{\alpha} f_\alpha(t)\,,$$

where $K(t)$ is the heat trace of $(\Gamma\backslash G, \mathbf{m})$. Here, α ranges over a discrete set. While we do not know whether this formula can be used to retrieve the length of smoothly closed geodesic, we will show that $\sum_{\alpha\neq 0} f_\alpha(t)$ is in $o(t^N)$ as $t \searrow 0$ for every $N \in \mathbb{N}$ (Proposition 1.82). Hence, f_0 determines the asymptotic behaviour of $K(t)$ as $t \searrow 0$ (Corollary 1.83). We then prove the rather surprising fact that all f_α are nonnegative (Corollary 1.84), which is not apparent from their definition. We also give a complete asymptotic expansion of $f_0(0)$ as $t \searrow 0$ in Lemma 1.90.

DEFINITION 1.72. A function $f : [0,\infty) \to \mathbb{R}$ is *completely monotone* if it lies in $C([0,\infty)) \cap C^\infty((0,\infty))$ and

$$(1.38) \qquad (-1)^n f^{(n)}(x) \geqslant 0 \quad \text{for all } n \in \mathbb{N}_0, x > 0\,.$$

We denote the set of all completely monotone functions by $\mathscr{CM}$.

REMARK 1.73. Note that (1.38) implies that a completely monotone function f and all its derivatives $f^{(n)}$, $n \in \mathbb{N}$, are bounded on $(0,\infty)$.

PROPOSITION 1.74 ([**SSV12**, Corollary 1.6]). *The set $\mathscr{CM}$ of completely monotone functions is a convex cone, i.e.,*

$$sf_1 + tf_2 \in \mathscr{CM} \quad \textit{for all } s,t \geqslant 0 \textit{ and } f_1, f_2 \in \mathscr{CM},$$

which is closed under multiplication, i.e.,

$$[0,\infty) \ni x \mapsto f_1(x)f_2(x) \in \mathbb{R} \quad \textit{lies in } \mathscr{CM} \textit{ for all } f_1, f_2 \in \mathscr{CM},$$

and under pointwise convergence.

LEMMA 1.75. *Let $\widetilde{\varphi} : [0,\infty) \ni x \mapsto \frac{\sqrt{x}}{\sinh\sqrt{x}} \in \mathbb{R}$. Then $\widetilde{\varphi} \in \mathscr{CM}$.*

PROOF. We consider the product expansion of the hyperbolic sine [**OLBC10**, 4.36.1]:

$$\sinh z = z \prod_{j=1}^{\infty} \frac{j^2\pi^2 + z^2}{j^2\pi^2}, \quad z \in \mathbb{C}\,.$$

Define for every $m \in \mathbb{N}$

$$\varphi_m : [0,\infty) \ni t \mapsto \prod_{j=1}^{m} \frac{j^2\pi^2}{j^2\pi^2 + t} \in \mathbb{R}\,.$$

Then clearly $\varphi_m \to \widetilde{\varphi}$ as $m \to \infty$ pointwise. Now each φ_m, $m \in \mathbb{N}$, is obviously completely monotone. As a pointwise limit of completely monotone functions, $\widetilde{\varphi}$ is completely monotone by Proposition 1.74. $\square$

LEMMA 1.76. *Let $\ell \in \mathbb{N}$. Consider the function $\varphi : \mathbb{R}^\ell \ni x \mapsto \|x\| \operatorname{csch} \|x\| \in \mathbb{R}$. Then φ lies in the Schwartz space $\mathscr{S}\left(\mathbb{R}^\ell\right)$. Moreoever, its Fourier transform $\widehat{\varphi} \in \mathscr{S}\left(\mathbb{R}^\ell\right)$ is nonnegative everywhere and has noncompact support.*

PROOF. We first prove that the function $\mathbb{R} \ni x \mapsto x \cdot \operatorname{csch} x \in \mathbb{R}$ lies in $\mathscr{S}(\mathbb{R})$. With the differentiation rules

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{csch} x = -\coth x \operatorname{csch} x \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x} \coth x = -\operatorname{csch}^2 x\,,$$

it is easy to see that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} x \operatorname{csch} x = \operatorname{csch} x \cdot P_n(x, \operatorname{csch} x, \coth x)\,,$$

where $P_n \in \mathbb{R}[v, w, z]$ is a polynomial. The left hand side has finite value in $x = 0$. Thus we have $\lim\limits_{x \to 0} P_n(x, \operatorname{csch} x, \coth x) = 0$. Since $\operatorname{csch} x$ decays exponentially for $x \to \pm\infty$, it follows that

$$\sup_{x \in \mathbb{R}} \left| x^\alpha \frac{\mathrm{d}^\beta}{\mathrm{d}x^\beta} x \operatorname{csch} x \right| < \infty\,,$$

for all $\alpha, \beta \in \mathbb{N}_0$, which proves that $x \mapsto x \cdot \operatorname{csch} x$ is a Schwartz function. Since $x \mapsto x \operatorname{csch} x$ is even, φ is smooth in $x = 0$ and hence lies in $\mathscr{S}\left(\mathbb{R}^\ell\right)$.

Denote by $\|\cdot\|_d$ the Euclidean norm on $\mathbb{R}^d$. By the Theorems of Schoenberg [**Fas07**, Theorem 5.2] and Bochner [**SSV12**, Theorem 4.14], a function $f : [0, \infty) \to \mathbb{R}$ is completely monotone if and only if $f \circ \|\cdot\|_d^2$ has nonnegative Fourier transform for all $d \in \mathbb{N}$. Hence, to prove that $\widehat{\varphi}$ is nonnegative everyhwere, it suffices to show that $\widetilde{\varphi} : [0, \infty) \ni x \mapsto \sqrt{x} \cdot \operatorname{csch} \sqrt{x} \in \mathbb{R}$ is completely monotone. This is the statement of Lemma 1.75.

To prove that $\widehat{\varphi}$ has noncompact support, suppose this was not the case, i.e., suppose $S := \operatorname{supp} \widehat{\varphi}$ is compact. Set $M := \sup\limits_{x \in S} |\widehat{\varphi}(x)|$ and define $f : \mathbb{C} \times \mathbb{R}^\ell \to \mathbb{C}$ by

$$(z, x) \mapsto e^{-2\pi i z \cdot x_1} \widehat{\varphi}(x)\,.$$

Then clearly $f(z, \cdot)$ is integrable on $\mathbb{R}^\ell$ for every $z \in \mathbb{C}$, because it is supported on the compact set S. Furthermore, $f(\cdot, x)$ is holomorphic for all $x \in \mathbb{R}^\ell$. At last, we obviously have $|f(z, \cdot)| \leqslant M$ on $\mathbb{R}^\ell$ for all $z \in \mathbb{C}$. It now follows from Theorem 1.38 that

$$F : \mathbb{C} \ni z \mapsto \int\limits_S f(z, x) \mathrm{d}\mu(x) \in \mathbb{C}$$

is an entire function. Since φ is even, we have $\hat{\hat{\varphi}} = \varphi$ (see Remark 1.33). It follows that

$$F(x) = \hat{\hat{\varphi}}(x \cdot e_1) = \varphi(x \cdot e_1) = |x| \cdot \operatorname{csch}|x| = x \cdot \operatorname{csch} x$$

for all $x \in \mathbb{R}$. The function $x \mapsto x \cdot \operatorname{csch} x$ is real analytic. Hence, $F(z) = z \cdot \operatorname{csch} z$ for all $z \in \mathbb{C}$. This is a contradiction to F being entire since $z \mapsto z \cdot \operatorname{csch} z$ has a pole at every $z = \pi i k$ with $k \in \mathbb{Z}\backslash\{0\}$. Thus, the support S of $\hat{\varphi}$ is not compact. □

REMARK 1.77. If $\ell = 1$ one can show that $\hat{\varphi} : \mathbb{R}^\ell \ni \omega \mapsto \frac{\pi^2}{2} \operatorname{sech}^2\left(\pi^2\omega\right) \in \mathbb{R}$ (see, e.g., [**EMOT54**, (18) in Chapter 1, Section 9]). In particular, $\hat{\varphi}$ is positive everywhere.

We will now prove our Poisson type formula. Let $(\Gamma\backslash G, \mathbf{m})$ be a compact nonsingular Heisenberg-like nilmanifold of dimension $\ell + 2n$, where ℓ is the dimension of the centre of G. We calculate the heat trace $K(t)$ of $(\Gamma\backslash G, \mathbf{m})$. By Corollary 1.14, we have

$$K(t) = \sum_{\sigma\in\Sigma(\Gamma\backslash G,\mathbf{m})} e^{-\sigma t} = \sum_{\sigma\in\Sigma_1(\Gamma\backslash G,\mathbf{m})} e^{-\sigma t} + \sum_{\sigma\in\Sigma_2(\Gamma\backslash G,\mathbf{m})} e^{-\sigma t}, \tag{1.39}$$

where

$$\sum_{\sigma\in\Sigma_1(\Gamma\backslash G,\mathbf{m})} e^{-\sigma t} = \sum_{\lambda\in\Gamma_{\mathfrak{n}}^*} e^{-4\pi^2\|\lambda\|_{\mathbf{m}_{\mathfrak{n}}}^2 t} =: \sum_{\lambda\in\Gamma_{\mathfrak{n}}^*} s_{H,t}^{\mathbf{m}}(\lambda) \tag{1.40}$$

and

$$\begin{aligned}
\sum_{\sigma\in\Sigma_2(\Gamma\backslash G,\mathbf{m})} e^{-\sigma t} &= \sum_{\lambda\in\Gamma_{\mathfrak{z}}^*\backslash\{0\}} \left(\prod_{j=1}^n c_j^{\mathbf{m}}\right) \|\lambda\|_{\mathbf{m}_{\mathfrak{z}}}^n \operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}} \sum_{p\in\mathbb{N}_0^n} e^{-\left(4\pi^2\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}}^2 + 2\pi\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}} \sum_{j=1}^n (2p_j+1)c_j^{\mathbf{m}}\right)t} \\
&= \operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}} \left(\prod_{j=1}^n c_j^{\mathbf{m}}\right) \sum_{\lambda\in\Gamma_{\mathfrak{z}}^*\backslash\{0\}} \|\lambda\|_{\mathbf{m}_{\mathfrak{z}}}^n e^{-4\pi^2\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}}^2 t + 2\pi\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}} \sum_{j=1}^n c_j^{\mathbf{m}} t} \prod_{j=1}^n \sum_{p_j\in\mathbb{N}_0} e^{-2\pi\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}} 2p_j c_j^{\mathbf{m}} t} \\
&= \operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}} \left(\prod_{j=1}^n c_j^{\mathbf{m}}\right) \sum_{\lambda\in\Gamma_{\mathfrak{z}}^*\backslash\{0\}} \|\lambda\|_{\mathbf{m}_{\mathfrak{z}}}^n e^{-4\pi^2\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}}^2 t + 2\pi\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}} \sum_{j=1}^n c_j^{\mathbf{m}} t} \prod_{j=1}^n \frac{1}{1 - e^{-4\pi\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}}^2 c_j^{\mathbf{m}} t}} \\
&= \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi t)^n} \sum_{\lambda\in\Gamma_{\mathfrak{z}}^*\backslash\{0\}} e^{-4\pi^2\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}}^2 t} \prod_{j=1}^n \frac{2\pi\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t\right)}
\end{aligned}$$

$$= \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi t)^n} \sum_{\lambda\in\Gamma_{\mathfrak{z}}^*\backslash\{0\}} s_{V,t}^{\mathbf{m}}(\lambda), \tag{1.41}$$

with

$$s_{V,t}^{\mathbf{m}} : \mathfrak{z} \ni \lambda \mapsto e^{-4\pi^2\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}}^2 t} \prod_{j=1}^n \frac{2\pi\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t\right)} \in \mathbb{R}. \tag{1.42}$$

THEOREM 1.78. *Let $(\Gamma\backslash G, \mathbf{m})$ be a compact nonsingular Heisenberg-like nilmanifold and $K(t)$ its heat trace. Then*

(i) $K(t) = \operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}} \sum_{X\in\Gamma_\mathfrak{n}\setminus\{0\}} \widehat{s^{\mathbf{m}}_{H,t}}(X) + \frac{\operatorname{Vol}(\Gamma\backslash G,\mathbf{m})}{(4\pi t)^n} \sum_{X\in\Gamma_\mathfrak{z}} \widehat{s^{\mathbf{m}}_{V,t}}(X)$,

(ii) $\widehat{s^{\mathbf{m}}_{H,t}}(X) = \frac{1}{(4\pi t)^n} e^{-\frac{\|X\|^2_{\mathbf{m}_\mathfrak{n}}}{4t}}$,

(iii) $\widehat{s^{\mathbf{m}}_{V,t}}(X) = \frac{1}{(4\pi t)^{\ell/2}} \sigma^{\mathbf{m}}_X(t)$,

where

$$\sigma^{\mathbf{m}}_X(t) := \pi^{-\ell/2} \int_\mathfrak{z} e^{-\frac{i}{\sqrt{t}}\langle X,\xi\rangle_{\mathbf{m}_\mathfrak{z}}} e^{-\|\xi\|^2_{\mathbf{m}_\mathfrak{z}}} \prod_{j=1}^n \frac{\sqrt{t} c^{\mathbf{m}}_j \|\xi\|_{\mathbf{m}_\mathfrak{z}}}{\sinh(\sqrt{t} c^{\mathbf{m}}_j \|\xi\|_{\mathbf{m}_\mathfrak{z}})} \,\mathrm{dVol}_\mathfrak{z}\,\xi\,. \tag{1.43}$$

Consequently

$$K(t) = \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi t)^n} \sum_{X\in\Gamma_\mathfrak{n}\setminus\{0\}} e^{-\frac{\|X\|^2}{4t}} + \frac{\operatorname{Vol}(\Gamma\backslash G,\mathbf{m})}{(4\pi t)^{\dim G/2}} \sum_{X\in\Gamma_\mathfrak{z}} \sigma^{\mathbf{m}}_X(t)\,. \tag{1.44}$$

PROOF. By (1.39), (1.40) and (1.41)

$$K(t) = \sum_{\lambda\in\Gamma^*_\mathfrak{n}} s^{\mathbf{m}}_{H,t}(\lambda) + \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi t)^n} \sum_{\lambda\in\Gamma^*_\mathfrak{z}\setminus\{0\}} s^{\mathbf{m}}_{V,t}(\lambda)\,. \tag{1.45}$$

An application of Poisson's summation formula (see Corollary 1.35) to the first term yields by Example 1.30 and Proposition 1.29*(vii)*

$$\begin{aligned}\sum_{\lambda\in\Gamma^*_\mathfrak{n}} s^{\mathbf{m}}_{H,t}(\lambda) &= \operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}} \sum_{X\in\Gamma_\mathfrak{n}} \widehat{s^{\mathbf{m}}_{H,t}}(X)\\ &= \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi t)^n} \sum_{X\in\Gamma_\mathfrak{n}\setminus\{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_\mathfrak{n}}}{4t}} + \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi t)^n}\,.\end{aligned}$$

A second application of Corollary 1.35, this time to the second term of (1.45), leads to

$$\begin{aligned}\frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi t)^n}\left(\sum_{\lambda\in\Gamma^*_\mathfrak{z}} s^{\mathbf{m}}_{V,t}(\lambda) - 1\right) &= \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi t)^n}\left(\operatorname{Vol} T_{\mathfrak{z},\mathbf{m}_\mathfrak{z}} \sum_{X\in\Gamma_\mathfrak{z}} \widehat{s^{\mathbf{m}}_{V,t}}(X) - 1\right)\\ &= \frac{\operatorname{Vol}(\Gamma\backslash G,\mathbf{m})}{(4\pi t)^n} \sum_{X\in\Gamma_\mathfrak{z}} \widehat{s^{\mathbf{m}}_{V,t}}(X) - \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi t)^n}\,.\end{aligned}$$

To complete the proof of the statement in the Theorem, we compute the Fourier transform of $s^{\mathbf{m}}_{V,t}$:

$$\begin{aligned}\widehat{s^{\mathbf{m}}_{V,t}}(X) &= \int_\mathfrak{z} e^{-2\pi i\langle X,\xi\rangle_{\mathbf{m}_\mathfrak{z}}} s^{\mathbf{m}}_{V,t}(\xi)\,\mathrm{dVol}_\mathfrak{z}(\xi)\\ &= \frac{\pi^{-\ell/2}}{(4\pi t)^{\ell/2}} \int_\mathfrak{z} e^{-\frac{i}{\sqrt{t}}\langle X,\xi\rangle_{\mathbf{m}_\mathfrak{z}}} e^{-\|\xi\|^2_{\mathbf{m}_\mathfrak{z}}} \prod_{j=1}^n \frac{\sqrt{t} c^{\mathbf{m}}_j \|\xi\|_{\mathbf{m}_\mathfrak{z}}}{\sinh(\sqrt{t} c^{\mathbf{m}}_j \|\xi\|_{\mathbf{m}_\mathfrak{z}})}\,\mathrm{dVol}_\mathfrak{z}(\xi) = \frac{1}{(4\pi t)^{\ell/2}} \sigma^{\mathbf{m}}_X(t)\,,\end{aligned}$$

where we have transformed the integral via $\mathfrak{z} \ni \xi \mapsto 2\pi\sqrt{t}\xi \in \mathfrak{z}$. $\square$

REMARK 1.79. By Theorem 1.78 and the calculations preceding it, we now have 2 different representations of the the heat trace $K(t)$ of a compact nonsingular Heisenberg-like nilmanifold:

$$\begin{aligned} K(t) &= \sum_{\lambda\in\Gamma_{\mathfrak{n}}^*} s^{\mathbf{m}}_{H,t}(\lambda) + \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi t)^n} \sum_{\lambda\in\Gamma_{\mathfrak{z}}^*\setminus\{0\}} s^{\mathbf{m}}_{V,t}(\lambda) \\ &= \operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}} \sum_{X\in\Gamma_{\mathfrak{n}}\setminus\{0\}} \widehat{s^{\mathbf{m}}_{H,t}}(X) + \frac{\operatorname{Vol}(\Gamma\backslash G,\mathbf{m})}{(4\pi t)^{\dim G/2}} \sum_{X\in\Gamma_{\mathfrak{z}}} \sigma^{\mathbf{m}}_X(t)\,. \end{aligned} \tag{1.46}$$

Both of these formulas consist of two terms. Note that these depend on different parts of the metric $\mathbf{m}$: in both lines, the first term depends only on $\mathbf{m}_{\mathfrak{n}}$, whereas the second term depends only on $\mathbf{m}_{\mathfrak{z}}$, $c^{\mathbf{m}}_1, \ldots, c^{\mathbf{m}}_n$ and the total volume $\operatorname{Vol}(\Gamma\backslash G,\mathbf{m}) = \operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}} \cdot \operatorname{Vol} T_{\mathfrak{z},\mathbf{m}_{\mathfrak{z}}}$.

REMARK 1.80. By introducing spherical coordinates, $\sigma^{\mathbf{m}}_0$ is easily seen to be given by the expression

$$\sigma^{\mathbf{m}}_0(t) = \pi^{-\ell/2} \operatorname{Vol}\left(S^{\ell-1}\right) \int_0^\infty e^{-r^2} \prod_{j=1}^n \frac{\sqrt{t}c^{\mathbf{m}}_j r}{\sinh(\sqrt{t}c^{\mathbf{m}}_j r)} r^{\ell-1} \mathrm{d}r\,. \tag{1.47}$$

LEMMA 1.81. *Let $f \in \mathscr{S}(\mathbb{R})$ and $P : \mathcal{P}_\ell \times \mathbb{R}^\ell \to \mathbb{R}$ be a polynomial. Then for every compact set $K \subset \mathcal{P}_\ell$ there exists a positive function $g \in L^1\left(\mathbb{R}^\ell\right)$ such that*

$$\left| f\left(\sqrt{A[\xi]}\right) \cdot P(A,\xi) \right| \leqslant g(\xi)$$

for all $A \in K$, $\xi \in \mathbb{R}^\ell$.

PROOF. If $P \equiv 0$ set $p = 0$. Otherwise, we view P as a polynomial in its second argument only and let $p \in \mathbb{N}_0$ be the corresponding degree, i.e.,

$$P(A,\xi) = \sum_{|\alpha|\leqslant p} a_\alpha(A)\xi^\alpha$$

where $a_\alpha : \mathcal{P}_\ell \to \mathbb{R}$, $\alpha \in \mathbb{N}_0^\ell$ with $|\alpha| \leqslant p$, are continuous functions. Since K is compact, the functions $|a_\alpha|_{|K} : K \to \mathbb{R}$ attain their respective maxima $\bar{a}_\alpha$ on K. By (1.16) we have

$$|P(A,\xi)| \leqslant \sum_{|\alpha|\leqslant p} |a_\alpha(A)|\,|\xi^\alpha| \leqslant \sum_{|\alpha|\leqslant p} \bar{a}_\alpha\,|\xi^\alpha| \leqslant \sum_{|\alpha|\leqslant p} \bar{a}_\alpha c_{\ell,\alpha}\,\|\xi\|^{|\alpha|} + 1 =: Q(\xi)\,, \tag{1.48}$$

for all $A \in K$ and $\xi \in \mathbb{R}^\ell$, where we have added 1 to ensure that Q is positive everywhere

The smallest eigenvalue λ_1 of $A \in \mathcal{P}_\ell$ is a continuous function $\lambda_1 : \mathcal{P}_\ell \to (0,\infty)$ (see Notation 1.61). Let $\underline{\lambda_1}$ be its minimum on K. Then we have $\underline{\lambda_1}\|\xi\|_2^2 = \underline{\lambda_1}\mathrm{Id}[\xi] \leqslant A[\xi]$ for all $A \in K$ and $\xi \in \mathbb{R}^\ell$. Since $f \in \mathscr{S}(\mathbb{R})$ there exists $C > 0$ such that

$$\sup_{x \in \mathbb{R}} \left| (1 + \underline{\lambda_1}^{-1/2}|x|)^{p+\ell+1} f(x) \right| \leqslant C\,,$$

which means that

$$|f(x)| \leqslant \frac{C}{(1 + \underline{\lambda_1}^{-1/2}|x|)^{p+\ell+1}}$$

for all $x \in \mathbb{R}$. Combining this inequality with (1.48) we obtain

$$\begin{aligned} \left| f\left(\sqrt{A[\xi]}\right) P(A,\xi) \right| &\leqslant \frac{C}{\left(1 + \underline{\lambda_1}^{-1/2}\sqrt{A[\xi]}\right)^{p+\ell+1}} |P(A,\xi)| \leqslant \frac{C \cdot Q(\xi)}{(1 + \underline{\lambda_1}^{-1/2}\underline{\lambda_1}^{1/2}\|\xi\|_2)^{p+\ell+1}} \\ &\leqslant \frac{C \cdot Q(\xi)}{(1 + \|\xi\|_2)^{p+\ell+1}} =: g(\xi) \end{aligned}$$

for all $A \in K$, $\xi \in \mathbb{R}^\ell$. The function g is positive by definition of Q. Since Q is a polynomial in $\|\xi\|$ of degree p, g is integrable. □

PROPOSITION 1.82. *Let G be a connected and simply connected nonsingular 2-step nilpotent Lie group and $\Gamma \subset G$ a uniform subgroup. Assume that $\mathscr{M}^{HL}(G) \neq \emptyset$ and let $K \subset \mathscr{M}^{HL}(G)$ be nonempty and compact and $\epsilon > 0$. Then*

(i) $\mathscr{M}^{HL}(G) \times (0,\infty) \ni (\mathbf{m}, t) \mapsto \sigma_X^{\mathbf{m}}(t) \in \mathbb{R}$ is continuous for every $X \in \Gamma_{\mathfrak{z}}$,

(ii) there exists a $C' > 0$ such that

$$\sum_{X \in \Gamma_{\mathfrak{z}} \setminus \{0\}} \sup_{(\mathbf{m},t) \in K \times [0,\epsilon]} |\sigma_X^{\mathbf{m}}(t)| \leqslant C'\,, \tag{1.49}$$

(iii) for every $N \in \mathbb{N}$ there exists $C > 0$ such that

$$\sum_{X \in \Gamma_{\mathfrak{z}} \setminus \{0\}} |\sigma_X^{\mathbf{m}}(t)| \leqslant C \cdot t^N \tag{1.50}$$

for all $t \in [0,\epsilon]$ and $\mathbf{m} \in K$.

Consequently, $t \mapsto \sigma_X^{\mathbf{m}}(t)$ is continuous on $[0,\infty)$ for every $X \in \Gamma_{\mathfrak{z}}$ and $\mathbf{m} \in \mathscr{M}^{HL}(G)$, the series

$$\sum_{X \in \Gamma_{\mathfrak{z}} \setminus \{0\}} \sigma_X^{\mathbf{m}}(t) \tag{1.51}$$

is locally normally convergent on $\mathscr{M}^{HL}(G) \times [0,\infty)$ and it lies, uniformly for every compact set $K \subset \mathscr{M}^{HL}(G)$, in $o(t^N)$ as $t \searrow 0$ for all $N \in \mathbb{N}$.

PROOF. Firstly, we choose a basis of $\mathfrak{z}$ and write $\xi \in \mathfrak{z}$ as $\xi = (\xi_1, \ldots, \xi_\ell)$ with respect to this basis. Also, we identify inner products $\mathbf{m}_{\mathfrak{z}}$ of $\mathfrak{z}$ with their matrix representations in $\mathcal{P}_\ell$ w.r.t. this basis. Under this identification, we have $\|\xi\|^2_{\mathbf{m}_{\mathfrak{z}}} = \mathbf{m}_{\mathfrak{z}}[\xi]$. Note that $\operatorname{csch} x =$

$\frac{1}{\sinh x}$. The function $\mathfrak{z} \ni \xi \mapsto \|\xi\|_{\mathbf{m}_\mathfrak{z}} \operatorname{csch} \|\xi\|_{\mathbf{m}_\mathfrak{z}} \in \mathbb{R}$ lies in $\mathscr{S}(\mathfrak{z})$ by Lemma 1.76. It follows that the function $\Phi(\mathbf{m}, t, \cdot)$ with

$$\Phi(\mathbf{m}, t, \xi) := e^{-\|\xi\|^2_{\mathbf{m}_\mathfrak{z}}} \prod_{j=1}^{n} \frac{\sqrt{t} c_j^{\mathbf{m}} \|\xi\|_{\mathbf{m}_\mathfrak{z}}}{\sinh(\sqrt{t} c_j^{\mathbf{m}} \|\xi\|_{\mathbf{m}_\mathfrak{z}})}$$

is in $\mathscr{S}(\mathfrak{z})$ for all $\mathbf{m} \in \mathscr{M}^{HL}(G)$ and $t \geqslant 0$. Let $\alpha \in \mathbb{N}_0^\ell$. Note that $\mathscr{S}(\mathfrak{z}) \subset L^1(\mathfrak{z})$. Hence, for any $\mathbf{m} \in \mathscr{M}^{HL}(G)$ and $t \geqslant 0$ we have

$$\left\| \frac{\partial^{|\alpha|}}{\partial \xi^\alpha} \Phi(\mathbf{m}, t, \xi) \right\|_{L^1(\mathfrak{z})} < \infty . \tag{1.52}$$

Note that the L^1-norm itself depends on $\mathbf{m}_\mathfrak{z}$ since the measure $\mathrm{dVol}_\mathfrak{z} = \sqrt{\det \mathbf{m}_\mathfrak{z}} d\xi$ depends on $\mathbf{m}_\mathfrak{z}$, but that the space $L^1(\mathfrak{z})$ does not depend on $\mathbf{m}_\mathfrak{z}$.

We will show that the left hand side of (1.52) is continuous in $(\mathbf{m}, t)$ and obtain intermediately *(i)*. Let $\varphi(x) := x \cdot \operatorname{csch} x$. The function φ is even and lies in $\mathscr{S}(\mathbb{R})$ (see Lemma 1.76). Hence, its derivative is odd and $\psi(x) := x^{-1}\varphi'(x)$ is even again and also lies in $\mathscr{S}(\mathbb{R})$. Thus, $\xi \mapsto \varphi\left(\sqrt{t} c_j^{\mathbf{m}} \|\xi\|_{\mathbf{m}_\mathfrak{z}}\right) \in \mathbb{R}$ is smooth and we have

$$\begin{aligned} \frac{\partial}{\partial \xi_i} \varphi\left(\sqrt{t} c_j^{\mathbf{m}} \|\xi\|_{\mathbf{m}_\mathfrak{z}}\right) &= \varphi'\left(\sqrt{t} c_j^{\mathbf{m}} \|\xi\|_{\mathbf{m}_\mathfrak{z}}\right) \frac{\sqrt{t} c_j^{\mathbf{m}}}{2\|\xi\|_{\mathbf{m}_\mathfrak{z}}} \left(\frac{\partial}{\partial \xi_i} \mathbf{m}_\mathfrak{z}[\xi]\right) \\ &= \psi\left(\sqrt{t} c_j^{\mathbf{m}} \|\xi\|_{\mathbf{m}_\mathfrak{z}}\right) \left(\sqrt{t} c_j^{\mathbf{m}}\right)^2 \frac{1}{2} \left(\frac{\partial}{\partial \xi_i} \mathbf{m}_\mathfrak{z}[\xi]\right) \end{aligned}$$

for every $1 \leqslant j \leqslant n$, $1 \leqslant i \leqslant \ell$ and $(\mathbf{m}, t) \in \mathscr{M}^{HL}(G) \times [0, \infty)$. Repeating this argument we have for every multi-index $\beta \in \mathbb{N}_0^\ell$:

$$\frac{\partial^{|\beta|}}{\partial \xi^\beta} \varphi\left(\sqrt{t} c_j^{\mathbf{m}} \|\xi\|_{\mathbf{m}_\mathfrak{z}}\right) = \sum_{k=0}^{|\beta|} \left(\sqrt{t} c_j^{\mathbf{m}}\right)^{2k} \psi_k\left(\sqrt{t} c_j^{\mathbf{m}} \|\xi\|_{\mathbf{m}_\mathfrak{z}}\right) \cdot P_k(\mathbf{m}_\mathfrak{z}, \xi) , \tag{1.53}$$

where ψ_k, $1 \leqslant k \leqslant |\beta|$, are even functions in $\mathscr{S}(\mathbb{R})$ and $P_k : \mathcal{P}_\ell \times \mathbb{R}^\ell \to \mathbb{R}$, $1 \leqslant k \leqslant |\beta|$, are polynomials. Let $K \subset \mathscr{M}^{HL}(G)$ be compact and $\epsilon > 0$. Since $c_j^{\mathbf{m}}$ is continuous in $\mathbf{m}$ (see Proposition 1.40), it attains its maximum $\overline{c_j}$ on K. From (1.53) we obtain that there exists a polynomial $Q_{|\beta|} : \mathcal{P}_\ell \times \mathbb{R}^\ell \to \mathbb{R}$ such that

$$\left| \frac{\partial^{|\beta|}}{\partial \xi^\beta} \varphi\left(\sqrt{t} c_j^{\mathbf{m}} \|\xi\|_{\mathbf{m}_\mathfrak{z}}\right) \right| \leqslant \sum_{k=0}^{|\beta|} \left(\sqrt{\epsilon} \overline{c_j}\right)^{2k} \max |\psi_k| \cdot |P_k(\mathbf{m}_\mathfrak{z}, \xi)| \leqslant \left|Q_{|\beta|}(\mathbf{m}_\mathfrak{z}, \xi)\right|$$

for all $t \in [0, \epsilon]$, $\mathbf{m} \in K$ and $\xi \in \mathbb{R}^\ell$. Consequently, there exists a polynomial $P : \mathcal{P}_\ell \times \mathbb{R}^\ell \to \mathbb{R}$ such that

$$\left| \frac{\partial^{|\alpha|}}{\partial \xi^\alpha} \Phi(\mathbf{m}, t, \xi) \right| \leqslant \left| e^{-\|\xi\|^2_{\mathbf{m}_\mathfrak{z}}} \cdot P(\mathbf{m}_\mathfrak{z}, \xi) \right| \tag{1.54}$$

for all $t \in [0,\epsilon]$, $\mathbf{m} \in K$ and $\xi \in \mathbb{R}^\ell$. This implies, by Lemma 1.81, that there exists a positive function $g \in L^1(\mathfrak{z})$ such that

$$\left|\frac{\partial^{|\alpha|}}{\partial \xi^\alpha}\Phi(\mathbf{m},t,\xi)\right| \leqslant g(\xi)$$

for all $t \in [0,\epsilon]$, $\mathbf{m} \in K$ and $\xi \in \mathbb{R}^\ell$. Note that the density $\sqrt{\det \mathbf{m}_\mathfrak{z}}$ attains a maximum on K. It follows that $K \times [0,\epsilon] \ni (\mathbf{m},t) \mapsto \sigma_X^{\mathbf{m}}(t) \in \mathbb{R}$ is continuous for every $X \in \Gamma_\mathfrak{z}$ by Theorem 1.36. By the same argument, it also follows that $\|\Phi(\mathbf{m},t,\cdot)\|_{L^1(\mathfrak{z})}$ is continuous in $(\mathbf{m},t) \in K \times [0,\epsilon]$. Hence

$$M_{\alpha,K,\epsilon} := \sup_{(\mathbf{m},t)\in K\times[0,\epsilon]} \left\|\frac{\partial^{|\alpha|}}{\partial \xi^\alpha}\Phi(\mathbf{m},t,\xi)\right\|_{L^1(\mathfrak{z})} = \max_{(\mathbf{m},t)\in K\times[0,\epsilon]} \left\|\frac{\partial^{|\alpha|}}{\partial \xi^\alpha}\Phi(\mathbf{m},t,\xi)\right\|_{L^1(\mathfrak{z})} < \infty\,.$$

Let $X \in \mathfrak{z}\backslash\{0\}$ and $N \in \mathbb{N}$. Choose $N' > \max\{\ell/2, N\}$. Since Schwartz functions vanish at infinity, integration by parts yields for every $1 \leqslant j \leqslant \ell$:

$$\frac{i}{\sqrt{t}}X_j \int_\mathfrak{z} e^{-\frac{i}{\sqrt{t}}\langle X,\xi\rangle_{\mathbf{m}_\mathfrak{z}}}\Phi(\mathbf{m},t,\xi)\;\mathrm{dVol}_\mathfrak{z}(\xi) = \int_\mathfrak{z} e^{-\frac{i}{\sqrt{t}}\langle X,\xi\rangle_{\mathbf{m}_\mathfrak{z}}}\frac{\partial}{\partial \xi_j}\Phi(\mathbf{m},t,\xi)\;\mathrm{dVol}_\mathfrak{z}(\xi)\,.$$

By iterating this step $2N'$ times and then taking absolute values, we obtain for every $1 \leqslant j \leqslant \ell$ and every $(\mathbf{m},t) \in K \times [0,\epsilon]$:

(1.55)

$$\frac{|X_j|^{2N'}}{t^{N'}}\left|\int_\mathfrak{z} e^{-\frac{i}{\sqrt{t}}\langle X,\xi\rangle_{\mathbf{m}_\mathfrak{z}}}\Phi(\mathbf{m},t,\xi)\;\mathrm{dVol}_\mathfrak{z}(\xi)\right| \leqslant \int_\mathfrak{z}\left|\frac{\partial^{2N'}}{\partial \xi_j^{2N'}}\Phi(\mathbf{m},t,\xi)\right|\mathrm{dVol}_\mathfrak{z}(\xi) \;\leqslant M_{2N'e_j,K,\epsilon}\,.$$

By taking the supremum, we also obtain the inequality

$$|X_j|^{2N'}\sup_{(\mathbf{m},t)\in K\times[0,\epsilon]}\left|\int_\mathfrak{z} e^{-\frac{i}{\sqrt{t}}\langle X,\xi\rangle_{\mathbf{m}_\mathfrak{z}}}\Phi(\mathbf{m},t,\xi)\;\mathrm{dVol}_\mathfrak{z}(\xi)\right| \leqslant \epsilon^{N'}M_{2N'e_j,K,\epsilon}\,. \tag{1.56}$$

We now sum in both (1.55) and (1.56) over $1 \leqslant j \leqslant \ell$ and $X \in \Gamma_\mathfrak{z}\backslash\{0\}$, which yields

$$\sum_{X\in\Gamma_\mathfrak{z}\backslash\{0\}}\left|\int_\mathfrak{z} e^{-\frac{i}{\sqrt{t}}\langle X,\xi\rangle_{\mathbf{m}_\mathfrak{z}}}\Phi(\mathbf{m},t,\xi)\;\mathrm{dVol}_\mathfrak{z}(\xi)\right| \leqslant t^{N'}\sum_{j=1}^{\ell}M_{2N'e_j,K,\epsilon}\sum_{X\in\Gamma_\mathfrak{z}\backslash\{0\}}\frac{1}{\|X\|_{2N'}^{2N'}}\,, \tag{1.57}$$

for all $\mathbf{m} \in K$ and $t \in [0,\epsilon]$ and

(1.58)

$$\sum_{X\in\Gamma_\mathfrak{z}\backslash\{0\}}\sup_{(\mathbf{m},t)\in K\times[0,\epsilon]}\left|\int_\mathfrak{z} e^{-\frac{i}{\sqrt{t}}\langle X,\xi\rangle_{\mathbf{m}_\mathfrak{z}}}\Phi(\mathbf{m},t,\xi)\;\mathrm{dVol}_\mathfrak{z}(\xi)\right| \leqslant \epsilon^{N'}\sum_{j=1}^{\ell}M_{2N'e_j,K,\epsilon}\sum_{X\in\Gamma_\mathfrak{z}\backslash\{0\}}\frac{1}{\|X\|_{2N'}^{2N'}}\,,$$

where $\|X\|_{2N'}^{2N'} = \sum_j |X_j|^{2N'}$.

The series on the right hand side of (1.57) and (1.58) converge for the following reason: Let $C'' > 0$ be such that $\|X\|_2 \leqslant C''\|X\|_{2N'}$ for all $X \in \mathfrak{z}$. Then we have for every $X \in \Gamma_{\mathfrak{z}}\backslash\{0\}$

$$\frac{1}{\|X\|_{2N'}^{2N'}} \leqslant C''^{2N'} \frac{1}{\|X\|_2^{2N'}}\,.$$

Let now M be a generator matrix for $\Gamma_{\mathfrak{z}}$; i.e., $M \in \mathrm{GL}(\ell;\mathbb{R})$ is such that $\Gamma_{\mathfrak{z}} = M \cdot \mathbb{Z}^{\ell}$. Furthermore, let $A := {}^{\mathrm{t}}MM$. Then

$$\sum_{X\in\Gamma_{\mathfrak{z}}\backslash\{0\}} \frac{1}{\|X\|_{2N'}^{2N'}} \leqslant C''^{2N'} \sum_{X\in\Gamma_{\mathfrak{z}}\backslash\{0\}} \frac{1}{\|X\|_2^{2N'}} = C''^{2N'} \sum_{m\in\mathbb{Z}^{\ell}\backslash\{0\}} \frac{1}{(A[m])^{N'}}\,. \tag{1.59}$$

The series on the right hand side of (1.59) is the defining Dirichlet series of Epstein's ζ-function applied to (A, N') (see Definition 2.33 on page 106). It converges by choice of N'.

From (1.57), (1.58) and (1.59) we get

$$\sum_{X\in\Gamma_{\mathfrak{z}}\backslash\{0\}} |\sigma_X^{\mathbf{m}}(t)| = \pi^{-\ell/2} \sum_{X\in\Gamma_{\mathfrak{z}}\backslash\{0\}} \left| \int_{\mathfrak{z}} e^{-\frac{i}{\sqrt{t}}\langle X,\xi\rangle_{\mathbf{m}_{\mathfrak{z}}}} \Phi(\mathbf{m},t,\xi)\ \mathrm{dVol}_{\mathfrak{z}}(\xi) \right| \leqslant \\ t^{N'}\pi^{-\ell/2}C''^{2N'} \sum_{j=1}^{\ell} M_{2N'e_j,K,\epsilon} \sum_{m\in\mathbb{Z}^{\ell}\backslash\{0\}} \frac{1}{(A[m])^{N'}}\,, \tag{1.60}$$

for every $(\mathbf{m}, t) \in K \times [0, \epsilon]$ and

$$\sum_{X\in\Gamma_{\mathfrak{z}}\backslash\{0\}} \sup_{(\mathbf{m},t)\in K\times[0,\epsilon]} |\sigma_X^{\mathbf{m}}(t)| \leqslant \epsilon^{N'}\pi^{-\ell/2}C''^{2N'} \sum_{j=1}^{\ell} M_{2N'e_j,K,\epsilon} \sum_{m\in\mathbb{Z}^{\ell}\backslash\{0\}} \frac{1}{(A[m])^{N'}}\,,$$

To obtain the statements of *(ii)* and *(iii)* define the constant C to be the right hand side of (1.60) divided by $t^{N'}$ and $C' := \epsilon^{N'} \cdot C$. □

Recall that the heat trace $K^{(M,g)}(t)$ of a compact Riemannian manifold (M, g) has an asymptotic expansion of the form:

$$K^{(M,g)}(t) \sim \frac{1}{(4\pi t)^{\dim M/2}} \left(a_0 + a_1 t + a_2 t^2 + \dots\right) \quad \text{as } t \searrow 0\,, \tag{1.61}$$

see, e.g., [**BGM71**, Chapitre III.E]. The a_i are spectral invariants called *heat invariants*.

COROLLARY 1.83. *Let $(\Gamma\backslash G, \mathbf{m})$ be as in Theorem 1.78 and $K(t)$ its heat trace. Then the asymptotic expansion*

$$K(t) \sim \frac{1}{(4\pi t)^{\dim G/2}} (a_0 + a_1 t + a_2 t^2 + \dots) \quad \textit{for} \quad t \searrow 0$$

is determined by $\sigma_0^{\mathbf{m}}$*, i.e.,*

$$K(t) - \frac{\operatorname{Vol}(\Gamma\backslash G, \mathbf{m})}{(4\pi t)^{\dim G/2}} \sigma_0^{\mathbf{m}}(t) \in o(t^N) \quad \text{as} \quad t \searrow 0 \quad \text{for all} \quad N \in \mathbb{N}_0 .$$

Consequently, $\sigma_0^{\mathbf{m}}$ *is infinitely often right-differentiable in* $t = 0$ *and we have*

$$K(t) \sim \frac{\operatorname{Vol}(\Gamma\backslash G, \mathbf{m})}{(4\pi t)^{\dim G/2}} \sum_{j=0}^{\infty} \frac{(\sigma_0^{\mathbf{m}})^{(j)}(+0)}{j!} t^j \quad \text{as} \quad t \searrow 0 .$$

PROOF. By Theorem 1.78 we have

$$K(t) = \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi t)^n} \sum_{X \in \Gamma_\mathfrak{n}\backslash\{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_\mathfrak{n}}}{4t}} + \frac{\operatorname{Vol}(\Gamma\backslash G, \mathbf{m})}{(4\pi t)^{\dim G/2}} \sum_{X \in \Gamma_\mathfrak{z}} \sigma_X^{\mathbf{m}}(t) .$$

Proposition 1.82(iii) says that $\frac{\operatorname{Vol}(\Gamma\backslash G,\mathbf{m})}{(4\pi t)^{\dim G/2}} \sum_{X \in \Gamma_\mathfrak{z}\backslash\{0\}} \sigma_X^{\mathbf{m}}(t)$ decays faster than any power of t as $t \searrow 0$. We will show that $\frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi t)^n} \sum_{X \in \Gamma_\mathfrak{n}\backslash\{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_\mathfrak{n}}}{4t}}$ also dacays faster than any power of t as $t \searrow 0$. The statement of the Corollary then follows from (1.61).

Let $f : \mathbb{R} \ni x \mapsto e^{-x^2} \in \mathbb{R}$. Then $f \in \mathscr{S}(\mathbb{R})$ by Example 1.22. Let $N \in \mathbb{N}$. Since $f \in \mathscr{S}(\mathbb{R})$ there exists $C > 0$ such that $|x^{2(N+n)} f(x)| < C$ for all $x \in \mathbb{R}$ and so in particular

$$f(x) < \frac{C}{x^{2(N+n)}} \qquad \text{for all } x \in (0, \infty) .$$

From this we obtain

$$\begin{aligned} \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi t)^n} \sum_{X \in \Gamma_\mathfrak{n}\backslash\{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_\mathfrak{n}}}{4t}} &= \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi t)^n} \sum_{X \in \Gamma_\mathfrak{n}\backslash\{0\}} f\left(\frac{\|X\|_{\mathbf{m}_\mathfrak{n}}}{2\sqrt{t}}\right) < \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi t)^n} \sum_{X \in \Gamma_\mathfrak{n}\backslash\{0\}} \frac{4^{N+n} \cdot C}{\|X\|^{2(N+n)}_{\mathbf{m}_\mathfrak{n}}} t^{N+n} \\ &= \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{\pi^n} \sum_{X \in \Gamma_\mathfrak{n}\backslash\{0\}} \frac{4^N \cdot C}{\|X\|^{2(N+n)}_{\mathbf{m}_\mathfrak{n}}} t^N . \end{aligned}$$

The series converges for the following reason: Let $(X_1, \ldots, X_n)$ be a basis of $\Gamma_\mathfrak{n}$ and $A \in \mathcal{P}_{2n}$ the associated Gram matrix, i.e., $\| \sum_j a_j X_j \|^2_{\mathbf{m}_\mathfrak{n}} = A[a]$ for all $a \in \mathbb{Z}^{2n}$. Then we have

$$\sum_{X \in \Gamma_\mathfrak{n}\backslash\{0\}} \|X\|^{-2(N+n)}_{\mathbf{m}_\mathfrak{n}} = \sum_{a \in \mathbb{Z}^{2n}\backslash\{0\}} \| \sum_j a_j X_j \|^{-2(N+n)}_{\mathbf{m}_\mathfrak{n}} = \sum_{a \in \mathbb{Z}^{2n}\backslash\{0\}} (A[a])^{-N-n} .$$

The right-hand side is the defining Dirichlet-series of Epstein's ζ-function, see Definition 2.33 on page 106. It converges since $n + N > n = 2n/2$. $\square$

Recall the definition of the function $s^{\mathbf{m}}_{V,t} : \mathfrak{z} \to \mathbb{R}$ in (1.42):

$$s_{V,t} : \mathfrak{z} \ni \lambda \mapsto e^{-4\pi^2 \|\lambda\|^2_{\mathbf{m}_\mathfrak{z}} t} \prod_{j=1}^{n} \frac{2\pi \|\lambda\|_{\mathbf{m}_\mathfrak{z}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi \|\lambda\|_{\mathbf{m}_\mathfrak{z}} c_j^{\mathbf{m}} t\right)} \in \mathbb{R} ,$$

The following is a corollary to Lemma 1.76.

COROLLARY 1.84. *The function $s^{\mathbf{m}}_{V,t} : \mathfrak{z} \to \mathbb{R}$ has positive Fourier transform for all $t > 0$. In particular, $\sigma^{\mathbf{m}}_X : [0,\infty) \to \mathbb{R}$ is positive on $(0,\infty)$ for every $X \in \Gamma_{\mathfrak{z}}$.*

PROOF. By Proposition 1.29*(xi)* we have

$$F[s^{\mathbf{m}}_{V,t}] = F\left[\lambda \mapsto e^{-4\pi^2\|\lambda\|^2_{\mathbf{m}_{\mathfrak{z}}} t}\right] \ast \mathop{\ast}_{j=1}^{n} F\left[\lambda \mapsto \frac{2\pi\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}} c^{\mathbf{m}}_j t}{\sinh(2\pi\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}} c^{\mathbf{m}}_j t)}\right],$$

where $\ast$ is the convolution defined in (1.19). The first function in this product is the Fourier transform of a Gaussian und thus a Gaussian (see Example 1.30 and Proposition 1.29*(vii)*); in particular, it is everywhere positive. According to Lemma 1.76 the other functions in this product are nonnegative and obviously nonzero. By the definition of the convolution (see (1.19)), the whole product is then a nonnegative function. Note that its support is all of $\mathfrak{z}$ since the support of the first factor is all of $\mathfrak{z}$.

Since $\sigma^{\mathbf{m}}_X(t) = (4\pi t)^{1/2}\widehat{s^{\mathbf{m}}_{V,t}}(X)$ by Theorem 1.78*(iii)*, we have $\sigma^{\mathbf{m}}_X(t) > 0$ for all $t > 0$. ☐

LEMMA 1.85. *For every $m \in \mathbb{N}_0$, the point $x = 0$ is a global extremum point of the function*

$$\psi_m : \mathbb{R} \ni x \mapsto \frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}} \prod_{j=1}^{n} \frac{c^{\mathbf{m}}_j x}{\sinh\left(c^{\mathbf{m}}_j x\right)} \in \mathbb{R}.$$

More precisely, $x = 0$ is a global minimum point if m is odd and $x = 0$ is a global maximum point if m is even. The global extremum is achieved only at $x = 0$.

PROOF. An argument similar to that in the last proof shows that ψ_0 has positive Fourier transform. More precisely: by Proposition 1.29*(xi)* one has

$$F[\psi_0] = \mathop{\ast}_{j=1}^{n} F\left[x \mapsto \frac{c^{\mathbf{m}}_j x}{\sinh\left(c^{\mathbf{m}}_j x\right)}\right],$$

the right hand side being a convolution of functions which are positive by Remark 1.77. Hence, $\widehat{\psi_0}(\omega) > 0$ for all $\omega \in \mathbb{R}$. The Fourier inversion (see Theorem 1.32) and Proposition 1.29*(ix)* give us for all $x \in \mathbb{R}$

$$\begin{aligned}\psi_m(x) = \frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}\psi_0(x) &= \int_{\mathbb{R}} e^{2\pi i x\xi}(2\pi i\xi)^{2m}\widehat{\psi_0}(\xi)\,\mathrm{d}\xi \\ &= (-1)^m 2(2\pi)^{2m}\int_0^\infty \cos(2\pi x\xi)\xi^{2m}\widehat{\psi_0}(\xi)\,\mathrm{d}\xi,\end{aligned}$$

where the last equality holds since $\widehat{\psi_0}$ is an even, real function. Since $\cos y < 1 = \cos 0$ almost everywhere and since $\widehat{\psi_0} > 0$, we arrive at

$$(-1)^m\psi_m(0) = 2(2\pi)^{2m}\int_0^\infty \xi^{2m}\widehat{\psi_0}(\xi)\,\mathrm{d}\xi > 2(2\pi)^{2m}\int_0^\infty \cos(2\pi x\xi)\xi^{2m}\widehat{\psi_0}(\xi)\,\mathrm{d}\xi$$
$$= (-1)^m\psi_m(x)\,,$$

for every $x \neq 0$, which proves the claim. □

For a point $x_0 \in \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{R}$ which is m-times differentiable in x_0, let $T^m_{x_0}[f](x)$ be the m-th degree Taylor polynomial of f at x_0, evaluated at $x \in \mathbb{R}$. In case f is only m-times right-differentiable in x_0, we denote the corresponding Taylor polynomial by $T^m_{+x_0}[f]$.

COROLLARY 1.86. *$(-1)^m\psi_0(x) \leqslant (-1)^m T^{2m}_0[\psi_0](x)$ for all $x \in \mathbb{R}$ and $m \in \mathbb{N}_0$, with equality if and only if $x = 0$.*

PROOF. Since ψ_0 and $T^{2m}_0[\psi_0]$ agree in $x = 0$, a sufficient condition for the claim to hold in all $x > 0$ is

$$(-1)^m\frac{\mathrm{d}}{\mathrm{d}x}\psi_0(x) < (-1)^m\frac{\mathrm{d}}{\mathrm{d}x}T^{2m}_0[\psi_0](x) = (-1)^m T^{2m-1}_0[\psi_0'](x)\,.$$

Proceeding inductively, we see that a sufficient condition for the statement to hold in all $x > 0$ is

$$\begin{aligned}(-1)^m\psi_m(x) &= \frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}\psi_0(x) < (-1)^m\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}T^{2m}_0[\psi_0](x) = (-1)^m T^0_0[\psi_m](x)\\ &= (-1)^m\psi_m(0)\,,\end{aligned}$$

which is guaranteed by Lemma 1.85. The statement for $x \leqslant 0$ now follows since ψ_0 and $T^{2m}_0[\psi_0]$ are even functions. □

REMARK 1.87. (i) The function ψ_0 is even and meromorphic. Hence, in $x = 0$ it is represented by a power series

$$\psi_0(x) = \sum_{j=0}^{\infty} a_j x^{2j}\,,$$

which has radius of convergence R for some $R > 0$. Replacing x by $\sqrt{x}$ yields a power series

$$\sum_{j=0}^{\infty} a_j x^j$$

with radius of convergence R^2. It follows that, after analytic continuation, the function

$$\widetilde{\psi_0}(x) := \psi_0(\sqrt{x})$$

is defined in a neighborhood of and analytic in $x = 0$. In particular, we have

$$T_0^{2m}[\psi_0](\sqrt{x}) = T_0^m\left[\widetilde{\psi_0}\right](x) \tag{1.62}$$

for all $x \geqslant 0$ and $m \in \mathbb{N}_0$.

(ii) Note that

$$\widetilde{\psi_0}(x) = \prod_{j=1}^{n} \widetilde{\varphi}\left((c_j^{\mathbf{m}})^2 x\right) \qquad \text{for all } x \geqslant 0\,,$$

where $\widetilde{\varphi}$ is the function defined in Lemma 1.75. It follows by Lemma 1.75, Proposition 1.74 and Remark 1.73 that for each $m \in \mathbb{N}_0$ and every set $K \subset \mathscr{M}(G)$ on which $c_n^{\mathbf{m}}$ is bounded, the function

$$K \times [0,\infty) \ni (\mathbf{m}, x) \mapsto \widetilde{\psi_0}^{(m)}(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\prod_{j=1}^{n} \widetilde{\varphi}\left((c_j^{\mathbf{m}})^2 x\right)\right) \in \mathbb{R}$$

is bounded.

Recall from Corollary 1.83 that $\sigma_0^{\mathbf{m}} : [0,\infty) \to \mathbb{R}$ is infinitely often continuously right differentiable in $t = 0$. Also, recall formula (1.47):

$$\sigma_0^{\mathbf{m}}(t) = \pi^{-\ell/2}\,\mathrm{Vol}\left(S^{\ell-1}\right)\int_0^\infty e^{-r^2}\psi_0(\sqrt{t}r)r^{\ell-1}\mathrm{d}r\,. \tag{1.63}$$

LEMMA 1.88. *We have for all $m \in \mathbb{N}_0$ and $t_0 \in [0,\infty)$:*

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}_{|t=t_0}\sigma_0^{\mathbf{m}}(t) = \pi^{-\ell/2}\,\mathrm{Vol}\left(S^{\ell-1}\right)\int_0^\infty e^{-r^2}\frac{\partial^m}{\partial t^m}_{|t=t_0}\psi_0(\sqrt{t}r)\cdot r^{\ell-1}\mathrm{d}r\,.$$

In particular, we have

$$T_{+0}^m[\sigma_0^{\mathbf{m}}](t) = \pi^{-\ell/2}\,\mathrm{Vol}\left(S^{\ell-1}\right)\int_0^\infty e^{-r^2}T_0^m[t \mapsto \psi_0(\sqrt{t}r)]\cdot r^{\ell-1}\mathrm{d}r \tag{1.64}$$

for all $t \geqslant 0$. Moreover, for any set $K \subset \mathcal{M}(G)$ on which $c_n^{\mathbf{m}}$ is bounded,

$$\mathscr{M}(G) \times [0,\infty) \ni (\mathbf{m}, t) \mapsto \frac{\mathrm{d}^m}{\mathrm{d}t^m}\sigma_0^{\mathbf{m}}(t) \in \mathbb{R}$$

is continuous and bounded on $K \times [0,\infty)$.

PROOF. By Remark 1.87(ii) there exists a $C_{m,K} > 0$ for every $m \in \mathbb{N}_0$ and every set $K \subset \mathscr{M}(G)$ on which $c_n^{\mathbf{m}}$ is bounded such that

$$e^{-r^2}\frac{\partial^m}{\partial t^m}\psi_0(\sqrt{t}r) = e^{-r^2}\frac{\partial^m}{\partial t^m}\widetilde{\psi_0}(tr^2) = e^{-r^2}\widetilde{\psi_0}^{(m)}(tr^2)\cdot r^{2m} \leqslant C_{m,K}\cdot e^{-r^2}r^{2m} \tag{1.65}$$

for all $r, t \in [0,\infty)$ and $\mathbf{m} \in K$. It follows from Theorem 1.37 that for any $t_0 \in [0,\infty)$:

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}_{|t=t_0} \sigma_0^{\mathbf{m}}(t) = \pi^{-\ell/2} \operatorname{Vol}\left(S^{\ell-1}\right) \int_0^\infty e^{-r^2} \frac{\partial^m}{\partial t^m}\Big|_{t=t_0} \psi_0(\sqrt{t}r) \cdot r^{\ell-1} \mathrm{d}r \,.$$

Formula (1.64) follows from Remark 1.87*(i)*. The continuity of $(\mathbf{m}, t) \mapsto (\sigma_0^{\mathbf{m}})^{(m)}(t)$ follows from (1.65) and Theorem 1.36 and boundedness from Remark 1.87(ii). $\square$

COROLLARY 1.89. *$(-1)^m \sigma_0^{\mathbf{m}}(t) \leqslant (-1)^m T_{+0}^m[\sigma_0^{\mathbf{m}}](t)$ for all $t \geqslant 0$ and $m \in \mathbb{N}_0$, with equality if and only if $t = 0$.*

PROOF. By Corollary 1.86 we have $(-1)^m \psi_0(x) \leqslant (-1)^m T_0^{2m}[\psi_0](x)$ for all $x \in \mathbb{R}$ and $m \in \mathbb{N}_0$, where "=" holds if and only if $x = 0$. Furthermore, by (1.62)

$$(-1)^m \psi_0(\sqrt{t}r) \leqslant (-1)^m T_0^{2m}[\psi_0](\sqrt{t}r) = (-1)^m T_0^m[t \mapsto \psi_0(\sqrt{t}r)](t)$$

for all $t, r \in [0,\infty)$ and $m \in \mathbb{N}_0$. Left and right hand side agree precisely when $t = 0$ or $r = 0$. The desired result now follows from (1.63) and (1.64). $\square$

LEMMA 1.90. *For $j \in \mathbb{N}_0$ define $b(j) := \frac{2(1-2^{2j-1})}{(2j)!} \cdot B_{2j}$, where B_j is the j-th Bernoulli number. Then we have for all $m \in \mathbb{N}_0$ and $t \geqslant 0$*

$$T_{+0}^m\left[\sigma_0^{\mathbf{m}}\right](t) = \sum_{j=0}^m a_j^{\mathbf{m}} t^j \tag{1.66}$$

with

$$a_j^{\mathbf{m}} := \pi^{-\ell/2} \operatorname{Vol}\left(S^{\ell-1}\right) \tfrac{1}{2}\Gamma(\tfrac{\ell}{2} + j) \sum_{\substack{j_1,\ldots,j_n \in \mathbb{N}_0 \\ j_1+\ldots+j_n = j}} \prod_{i=1}^n b(j_i)(c_i^{\mathbf{m}})^{2j_i} \,. \tag{1.67}$$

In particular, since the sign of $b(j)$ is $(-1)^j$, every term in the coefficient of t^j has sign $(-1)^j$.

PROOF. First consider the case $n = 1$. We have

$$T_0^m[t \mapsto c_1^{\mathbf{m}}\sqrt{t}r \operatorname{csch}(c_1^{\mathbf{m}}\sqrt{t}r)](t) = T_0^{2m}[x \mapsto x \operatorname{csch} x](c_1^{\mathbf{m}}\sqrt{t}r)$$

which, by [**OLBC10**, 4.19.4], equals

$$\sum_{j=0}^m \frac{2\left(1-2^{2j-1}\right)B_{2j}}{(2j)!}(c_1^{\mathbf{m}})^{2j} t^j r^{2j} = \sum_{j=0}^m b(j)(c_1^{\mathbf{m}})^{2j} t^j r^{2j} \,.$$

Lemma 1.88 now gives us

$$T_{+0}^m[\sigma_0^{\mathbf{m}}](t) = \pi^{-\ell/2} \operatorname{Vol} S^{\ell-1} \int_0^\infty e^{-r^2} T_0^m\left[t \mapsto c_1^{\mathbf{m}}\sqrt{t}r \operatorname{csch}\left(c_1^{\mathbf{m}}\sqrt{t}r\right)\right](t) r^{\ell-1} \mathrm{d}r$$

$$= \pi^{-\ell/2} \operatorname{Vol} S^{\ell-1} \sum_{j=0}^{m} b(j)(c_1^{\mathbf{m}})^{2j} t^j \int_0^\infty e^{-r^2} r^{2j+\ell-1} \mathrm{d}r$$

$$= \pi^{-\ell/2} \operatorname{Vol} S^{\ell-1} \sum_{j=0}^{m} b(j)(c_1^{\mathbf{m}})^{2j} t^j \tfrac{1}{2}\Gamma\left(\tfrac{\ell}{2}+j\right),$$

where we have used the integral representation of the gamma function(see [**OLBC10**, 5.9.1]). The formula for $n > 1$ follows from an application of the Cauchy product formula. □

REMARK 1.91. Note that $\sigma_0^{\mathbf{m}}$ determines the asymptotic expansion of the heat trace (see Corollary 1.83). This means that we have just computed all heat invariants of Heisenberg-like nilmanifolds.

REMARK 1.92. We present an argument that will assure the convergence and asymptotic behaviour of certain recurring series. Let $f \in \mathscr{S}(\mathbb{R})$, $L \subset (V, \langle\cdot,\cdot\rangle)$ a lattice of full rank in the n-dimensional real vector space V and $\alpha, \beta > 0$. Denote the set of inner products on V by $\mathscr{M}(V)$. For $t > 0$ and $\mathbf{m} \in \mathscr{M}(V)$ consider the series $\sum_{v\in L\setminus\{0\}} f\left(\|v\|_{\mathbf{m}}^{\alpha} t^{\beta}\right)$. Let $N \in \mathbb{N}$ be such that $N\alpha > n$. By Definition 1.19 there exists $C_N > 0$ such that

$$|f(x)| \leqslant \frac{C_N}{|x|^N} \text{ for all } x \in \mathbb{R}\setminus\{0\}.$$

Hence,

$$\sum_{v\in L\setminus\{0\}} \left| f\left(\|v\|_{\mathbf{m}}^{\alpha} t^{\beta}\right)\right| \leqslant C_N \sum_{v\in L\setminus\{0\}} \|v\|_{\mathbf{m}}^{-\alpha N} \cdot t^{-\beta N}. \tag{1.68}$$

Let $(v_1, \ldots, v_n)$ be a basis for L and $A_{\mathbf{m}} \in \mathcal{P}_n$ the corresponding Gram matrix in the metric $\mathbf{m}$, i.e., $(A_{\mathbf{m}})_{i,j} = \mathbf{m}(v_i, v_j)$. Then we have

$$\sum_{v\in L\setminus\{0\}} \|v\|_{\mathbf{m}}^{-\alpha N} = \sum_{a\in\mathbb{Z}^n\setminus\{0\}} (A_{\mathbf{m}}[a])^{-\alpha N/2}.$$

The right hand side is the defining Dirichlet series of Epstein's ζ-function (see Definition 2.33 on page 106). It converges since N was chosen such that $\alpha N/2 > n/2$. Moreover, w.r.t. $A_{\mathbf{m}}$ it converges locally normally, see Remark 2.34. Hence, for every compact set $K \in \mathscr{M}(V)$ and every $\epsilon > 0$ we have that the series

$$K \times [\epsilon, \infty) \ni (\mathbf{m}, t) \mapsto \sum_{v\in L\setminus\{0\}} f\left(\|v\|_{\mathbf{m}}^{\alpha} t^{\beta}\right) = \sum_{a\in\mathbb{Z}^n\setminus\{0\}} f\left((A_{\mathbf{m}}[a])^{\alpha/2} t^{\beta}\right) \in \mathbb{R} \tag{1.69}$$

converges normally. Furthermore, it is, uniformly on K, in $o(t^{-N})$ as $t \to \infty$ for every $N \in \mathbb{N}$.

Replacing t by $1/t$ in (1.69) yields a normally convergent series

$$K \times [0,\epsilon] \ni (\mathbf{m}, t) \mapsto \sum_{v \in L\setminus\{0\}} f\left(\|v\|_{\mathbf{m}}^{\alpha} t^{-\beta}\right) \in \mathbb{R} \tag{1.70}$$

which is, uniformly on K, in $o(t^N)$ as $t \searrow 0$ for every $N \in \mathbb{N}$.

Note that we have already used above argument in the proofs of Proposition 1.82 and Corollary 1.83.

We would like to apply the last remark to the series

$$\frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi t)^n} \sum_{X \in \Gamma_{\mathfrak{n}}\setminus\{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_{\mathfrak{n}}}}{4t}}, \quad \sum_{\lambda \in \Gamma_{\mathfrak{n}}^*\setminus\{0\}} e^{-4\pi^2\|\lambda\|^2_{\mathbf{m}_{\mathfrak{n}}} t}, \qquad \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi t)^n} \sum_{\lambda \in \Gamma_{\mathfrak{z}}^*\setminus\{0\}} s_{V,t}^{\mathbf{m}}(\lambda)\,. \tag{1.71}$$

However, by changing the metric $\mathbf{m} \in \mathscr{M}^{HL}(G)$, the lattices $\Gamma_{\mathfrak{z}}^*$, $\Gamma_{\mathfrak{n}}$ and $\Gamma_{\mathfrak{n}}^*$ also change. Even worse, the subspace $\mathfrak{n} \subset \mathfrak{g}$ depends on the metric $\mathbf{m}$. Hence, it is a priori not clear what normal convergence of one of the series in (1.71) means. We work around this by using coordinates. Let $\mathfrak{B} = (Z_1, \dots, Z_\ell, X_1, \dots, X_{2n})$ be a basis of $\mathfrak{g}$ as in Proposition 1.8. We identify an inner product $\mathbf{m} \in \mathscr{M}^{HL}(G)$ of $\mathfrak{g}$ with its Gram matrix in $\mathcal{P}_{\ell+2n}$. Likewise we view $\mathbf{m}_{\mathfrak{z}}$ and $\mathbf{m}_{\mathfrak{n}}$ as matrices in $\mathcal{P}_\ell$ and $\mathcal{P}_{2n}$, respectively. Let $\pi_{\mathfrak{z}}$ and $\pi_{\mathfrak{n}}$ be the respective orthogonal projections onto $\mathfrak{z}$ and $\mathfrak{n}$. We view these as functions $\pi_{\mathfrak{z}}, \pi_{\mathfrak{n}} : \mathfrak{g} \times \mathscr{M}^{HL}(G) \to \mathfrak{g}$ to make the dependence on the metric explicit. Let $X = \sum_{j=1}^{\ell} \alpha_j Z_j + \sum_{k=1}^{2n} \beta_k X_k$. Then $X \mapsto \pi_{\mathfrak{z}}(X, \mathbf{m})$ being an orthogonal projection onto $\mathfrak{z}$ is equivalent to

$$\pi_{\mathfrak{z}}(X, \mathbf{m}) = \sum_{j=1}^{\ell} \pi_j(X, \mathbf{m}) Z_j\,,$$
$$\langle \pi_{\mathfrak{z}}(X, \mathbf{m}) - X, Z_i \rangle_{\mathbf{m}} = 0 \quad \text{for all } i = 1, \dots, \ell\,,$$

where $\pi_j : \mathfrak{g} \times \mathscr{M}^{HL}(G) \to \mathfrak{g}$, $j = 1, \dots, \ell$, are functions linear in their first argument. Substituting the first equation into the second yields

$$\sum_{j=1}^{\ell} \pi_j(X, \mathbf{m}) \langle Z_j, Z_i \rangle_{\mathbf{m}} = \langle X, Z_i \rangle_{\mathbf{m}} \quad \text{for all } i = 1, \dots, \ell\,. \tag{1.72}$$

Denote by $\tilde{X}_{\mathbf{m}}$ the vector with components $\langle X, Z_i \rangle_{\mathbf{m}}$, $i = 1, \dots, \ell$ and identify $\pi_{\mathfrak{z}}(X, \mathbf{m})$ with its representation w.r.t. the basis $(Z_1, \dots, Z_\ell)$. Then (1.72) is equivalent to $\mathbf{m}_{\mathfrak{z}} \cdot \pi_{\mathfrak{z}}(X, \mathbf{m}) = \tilde{X}_{\mathbf{m}}$, i.e.,

$$\pi_{\mathfrak{z}}(X, \mathbf{m}) = \mathbf{m}_{\mathfrak{z}}^{-1} \cdot \tilde{X}_{\mathbf{m}}\,,$$

which is clearly continuous in $\mathbf{m}$. On the other hand, we have $\pi_{\mathfrak{n}}(X, \mathbf{m}) = X - \pi_{\mathfrak{z}}(X, \mathbf{m})$ for all $X \in \mathfrak{g}$, $\mathbf{m} \in \mathscr{M}^{HL}(G)$. Hence, $\pi_{\mathfrak{n}}$ is continuous too. By Proposition 1.9,

$(\pi_{\mathfrak{n}}(X_1), \ldots, \pi_{\mathfrak{n}}(X_{2n}))$ is a basis of $\Gamma_{\mathfrak{n}}$. It follows that $\mathbf{m} \mapsto \mathbf{m}_{\mathfrak{n}} \in \mathcal{P}_{2n}$, $(\mathbf{m}_{\mathfrak{n}})_{i,j} = \langle \pi_{\mathfrak{n}}(X_i), \pi_{\mathfrak{n}}(X_j) \rangle_{\mathbf{m}}$, is continuous. Now we clearly have

$$\frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi t)^n} \sum_{X \in \Gamma_{\mathfrak{n}} \setminus \{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_{\mathfrak{n}}}}{4t}} = \frac{\sqrt{\det \mathbf{m}_n}}{(4\pi t)^n} \sum_{a \in \mathbb{Z}^{2n} \setminus \{0\}} e^{-\frac{\mathbf{m}_{\mathfrak{n}}[a]}{4t}} \tag{1.73}$$

and

$$\sum_{\lambda \in \Gamma^*_{\mathfrak{n}} \setminus \{0\}} e^{-4\pi^2 \|\lambda\|^2_{\mathbf{m}_{\mathfrak{n}}} t} = \sum_{a \in \mathbb{Z}^{2n} \setminus \{0\}} e^{-4\pi^2 \mathbf{m}_{\mathfrak{n}}^{-1}[a] t}\,. \tag{1.74}$$

By the above, each term on the right-hand side of (1.73) and (1.74) is a continuous function of $\mathbf{m} \in \mathscr{M}^{HL}(G)$. Analogously, we have

(1.75)

$$\frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi t)^n} \sum_{\lambda \in \Gamma^*_{\mathfrak{z}} \setminus \{0\}} s^{\mathbf{m}}_{V,t}(\lambda) = \frac{\sqrt{\det \mathbf{m}_n}}{(4\pi t)^n} \sum_{a \in \mathbb{Z}^{\ell} \setminus \{0\}} e^{-4\pi^2 \mathbf{m}_{\mathfrak{z}}^{-1}[a] t} \prod_{j=1}^{n} \frac{2\pi \sqrt{\mathbf{m}_{\mathfrak{z}}^{-1}[a]}\, c_j^{\mathbf{m}} t}{\sinh\left(2\pi \sqrt{\mathbf{m}_{\mathfrak{z}}^{-1}[a]}\, c_j^{\mathbf{m}} t\right)}\,.$$

Here, too, every term on the right hand side of (1.75) is a continuous function of $\mathbf{m} \in \mathscr{M}^{HL}(G)$. With these formulas, we can apply Remark 1.92 to the series (1.71). We summarise the statements in the following proposition.

PROPOSITION 1.93. *Let $K \subset \mathscr{M}^{HL}(G)$ be compact. Then*

(i) $K \times [0,1] \ni (\mathbf{m}, t) \mapsto \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi t)^n} \sum_{X \in \Gamma_{\mathfrak{n}} \setminus \{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_{\mathfrak{n}}}}{4t}}$ *converges normally,*

(ii) $K \times [1,\infty) \ni (\mathbf{m}, t) \mapsto \sum_{\lambda \in \Gamma^*_{\mathfrak{n}} \setminus \{0\}} e^{-4\pi^2 \|\lambda\|^2_{\mathbf{m}_{\mathfrak{n}}} t}$ *converges normally and*

(iii) $K \times [1,\infty) \ni (\mathbf{m}, t) \mapsto \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi t)^n} \sum_{\lambda \in \Gamma^*_{\mathfrak{z}} \setminus \{0\}} s^{\mathbf{m}}_{V,t}(\lambda)$ *converges normally.*

Furthermore, for every $N \in \mathbb{N}$ there exist $C_i > 0$, $i = 0, 1, 2$, such that

(a) $\frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi t)^n} \sum_{X \in \Gamma_{\mathfrak{n}} \setminus \{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_{\mathfrak{n}}}}{4t}} < C_0 \cdot t^N$ *for all $t \in [0,1]$ and all $\mathbf{m} \in K$,*

(b) $\sum_{\lambda \in \Gamma^*_{\mathfrak{n}} \setminus \{0\}} e^{-4\pi^2 \|\lambda\|^2_{\mathbf{m}_{\mathfrak{n}}} t} < C_1 \cdot t^{-N}$ *for all $t \in [1,\infty)$ and all $\mathbf{m} \in K$, and*

(c) $\frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi t)^n} \sum_{\lambda \in \Gamma^*_{\mathfrak{z}} \setminus \{0\}} s^{\mathbf{m}}_{V,t}(\lambda) < C_2 \cdot t^{-N}$ *for all $t \in [1,\infty)$ and all $\mathbf{m} \in K$.*

REMARK 1.94. Theorem 1.78 was not conceived of nothing. The author's starting point for it was a Poisson formula for normalised Heisenberg manifolds by H. Pesce **[Pes94]**. If $K(t)$ is the heat trace of the normalised Heisenberg manifold $(\Gamma \backslash G, \mathbf{m}) = (\Gamma^r \backslash H_n, \mathbf{m})$, then that formula reads

$$K(t) = \frac{\operatorname{Vol}(\Gamma \backslash G)}{(4\pi t)^{\dim G/2}} \sum_{\alpha} f_\alpha(t) e^{-\frac{\alpha^2}{4t}}\,. \tag{1.76}$$

Here, α ranges over the set of lengths of smoothly closed geodesics of $(\Gamma\backslash G, \mathbf{m})$. The functions f_α have at most a pole at $t = +0$ so that f_0 determines the asymptotic expansion of $K(t)$ as $t \searrow 0$. Moreover, an analysis of the full asymptotic expansion of $f_\alpha(t)$ as $t \searrow 0$ allowed H. Pesce to show that the length spectrum of $(\Gamma\backslash G, \mathbf{m})$ (without multiplicities) can be recovered from the spectrum of $(\Gamma\backslash G, \mathbf{m})$.

In his unpublished thesis **[Bla98]**, H. Blanchard found a formula analogous to (1.76) for all Heisenberg type nilmanifolds $(\Gamma\backslash G, \mathbf{m})$, where G has odd dimensional centre. With his formula he was also able to read off the length spectrum from the Laplace spectrum.

While working on the height of compact nilmanifolds, the author generalised H. Blanchard's formula to all nonsingular Heisenberg-like nilmanifolds $(\Gamma\backslash G, \mathbf{m})$, where G has odd dimensional centre. The wish to remove the additional assumption on the parity of the dimension of the centre of G then resulted in Theorem 1.78. Unfortunately, the application of the Poisson formula to the retrieval of the length spectrum failed which is why we omit the result.

CHAPTER 2

ζ-Functions and Heights

This chapter contains the main results of this thesis. In section 1 we introduce the central object, the spectral ζ-function of a Riemannian manifold. We obtain formulas for its meromorphic continuation and for the height in the case of a compact Heisenberg-like nilmanifold. Section 2 explores when the ζ-function and the height are bounded from below as functions on the moduli space of Heisenberg-like metrics. Lastly, in section 3, we present work by P. Sarnak et al. on the spectral ζ-function and the height of flat tori and identify a few extremal metrics for the height of normalised Heisenberg manifolds.

1. Formulas for the ζ-Function and the Height

DEFINITION 2.1.

(i) For $\alpha \in \mathbb{R}$ define the set

$$D(\alpha) := \{s \in \mathbb{C} \mid \Re s > \alpha\} \ . \tag{2.1}$$

$D(\alpha)$ is the right half plane with axis of abscissa $\Re s = \alpha$.

(ii) For $k \in \mathbb{N}$ define

$$\operatorname{sing}(k) := \left(\tfrac{k}{2} - \mathbb{N}_0\right) \setminus (-\mathbb{N}_0) \, . \tag{2.2}$$

If k is odd we have $\operatorname{sing}(k) = \{k/2, k/2 - 1, \ldots\}$, whereas if k is even we have $\operatorname{sing}(k) = \{k/2, k/2 - 1, \ldots, 1\}$.

(iii) For any compact Riemannian manifold (M, g) define the spectral ζ-function $\zeta((M, g), s)$ of (M, g) to be the ζ-function associated with the spectrum of the Laplace-Beltrami operator Δ of (M, g), i.e.,

$$\mathbb{C} \ni s \mapsto \zeta((M, g)) = \sum_{\lambda > 0} \lambda^{-s} \in \mathbb{C} \, . \tag{2.3}$$

Here, the sum extends over all nonzero eigenvalues of Δ and the branch of λ^s is chosen so that $\lambda^s > 0$ if $\lambda > 0$ and $s \in \mathbb{R}$. By Weyl's law, the sum in (2.3) converges absolutely for $s \in D(\dim M/2)$ and $\zeta((M, g), s)$ has a meromorphic continuation to all of $\mathbb{C}$ with at most simple poles located at $\operatorname{sing}(\dim M)$ (see, e.g., **[See67]**, **[MP49]**).

(iv) For ζ-functions of any kind, not just the spectral ζ-functions defined in *(iii)*, we will denote the s-derivative by ζ'. Thus, e.g.,

$$\zeta'((M,g),s) = \frac{\partial}{\partial s}\zeta((M,g),s)\,.$$

Recall the definition of a proper metric space M, i.e., a metric space in which every closed ball is compact. In particular, every point $p \in M$ has a compact neighborhood.

LEMMA 2.2. *Let M be a proper metric space and $\alpha \in \mathbb{R}$. Furthermore, let $f : M \times (0,1] \to \mathbb{R}$ be a continuous function such that for every compact set $K \subset M$ and every $N \in \mathbb{N}$ there exists a $C > 0$ such that*

$$|f(x,t)| \leqslant C \cdot t^N \quad \textit{for all } x \in K, t \in (0,1]\,.$$

The function

$$F : M \times \mathbb{C} \ni (x,s) \mapsto \int_0^1 f(x,t) \cdot t^{s-\alpha} \mathrm{d}t \in \mathbb{C}$$

is continuous and $\mathbb{C} \ni s \mapsto F(x,s) \in \mathbb{C}$ is holomorphic for every $x \in M$.

PROOF. Let $G : M \times \mathbb{C} \times (0,1] \ni (x,s,t) \mapsto f(x,t) \cdot t^{s-\alpha} \to \mathbb{C}$. Clearly, G is continuous and $s \mapsto G(x,s,t)$ is holomorphic for every $x \in M$ and $t \in (0,1]$. Let $K \subset M$ be compact and $D \subset \mathbb{C}$ a compact disc. Choose $N \in \mathbb{N}$ such that $N > \alpha - 1 - \Re s$ for all $s \in D$ and let $\delta > 0$ be such that $N > \alpha - 1 - \Re s + \delta$ for all $s \in D$. By the assumption on f there exists $C > 0$ such that

$$|G(x,s,t)| = \left|f(x,t) \cdot t^{s-\alpha}\right| \leqslant C \cdot t^{N+\Re s-\alpha} < C \cdot t^{-1+\delta} \tag{2.4}$$

for all $x \in K$ and $s \in D$. The right hand side of (2.4) is integrable on $(0,1]$. The statement of the Lemma now follows from Theorems 1.36 and 1.38. ☐

We can dualize the last lemma in the following sense:

LEMMA 2.3. *Let M be a proper metric space and $\alpha \in \mathbb{R}$. Furthermore, let $f : M \times [1,\infty) \to \mathbb{R}$ be a continuous function such that for every compact set $K \subset M$ and every $N \in \mathbb{N}$ there exists a $C > 0$ such that*

$$|f(x,t)| \leqslant C \cdot t^{-N} \quad \textit{for all } x \in K, t \in [1,\infty)\,.$$

The function

$$F : M \times \mathbb{C} \ni (x,s) \mapsto \int_1^\infty f(x,t) \cdot t^{s-\alpha} \mathrm{d}t \in \mathbb{C}$$

is continuous and $\mathbb{C} \ni s \mapsto F(x,s) \in \mathbb{C}$ is holomorphic for every $x \in M$.

PROOF. We define $\tilde{f}(x,t) := f(x, 1/t)$. Then the last Lemma tells us that

$$\tilde{F}(x,s) := \int_0^1 \tilde{f}(x,t) \cdot t^{s-\alpha} \mathrm{d}t = \int_0^1 f(x, 1/t) \cdot t^{s-\alpha} \mathrm{d}t$$

is continuous and holomorphic in s for every $x \in M$. We transform the integral via the diffeomorphism $(0,1] \ni t \mapsto 1/t \in [1,\infty)$ and obtain

$$\tilde{F}(x,s) = \int_1^\infty f(x,t) \cdot t^{\alpha-s-2} \mathrm{d}t = F(x, -s+2\alpha+2)\,.$$

Since $\rho \mapsto -\rho + 2\alpha + 2$ is holomorphic, the claim follows. $\square$

Let $(\Gamma\backslash G, \mathbf{m})$ be a compact nonsingular Heisenberg-like nilmanifold. Recall the definition of $s^{\mathbf{m}}_{V,t}$ in (1.42) and the definition of $\sigma^{\mathbf{m}}_X$ in (1.43). Also, recall from Corollary 1.83 that $\sigma^{\mathbf{m}}_0 : [0,\infty) \to \mathbb{R}$ is infinitely often continuously right-differentiable in $t = 0$. Denote by $T^N_{+0}[\sigma^{\mathbf{m}}_0](t) = \sum_{j=0}^N a^{\mathbf{m}}_j t^j$ the N-th degree Taylor polynomial of $\sigma^{\mathbf{m}}_0$ in $t_0 = +0$ evaluated at t. Furthermore, recall either from Lemma 1.88 or from the explicit formula for $T^N_{+0}[\sigma^{\mathbf{m}}_0](t)$ from Lemma 1.90 and Proposition 1.40*(ii)* that the coefficients $a^{\mathbf{m}}_j$ are continuous when considered as functions on the space $\mathscr{M}^{HL}(G)$ of Heisenberg-like metrics on G: $\mathscr{M}^{HL}(G) \ni \mathbf{m} \mapsto (-1)^j a^{\mathbf{m}}_j \in (0,\infty)$.

LEMMA 2.4. *Let $N \in \mathbb{N}_0 \cup \{-1\}$ and $T^N_{+0}[\sigma^{\mathbf{m}}_0](t) = \sum_{j=0}^N a^{\mathbf{m}}_j t^j$. The function*

$$(\mathbf{m},s) \mapsto \frac{1}{\Gamma(s)} \frac{\operatorname{Vol}(\Gamma\backslash G, \mathbf{m})}{(4\pi)^{\dim G/2}} \left(\int_0^1 \left(\sigma^{\mathbf{m}}_0(t) - T^N_{+0}[\sigma^{\mathbf{m}}_0](t) \right) t^{s-\dim G/2-1} \mathrm{d}t + \sum_{j=0}^N \frac{a^{\mathbf{m}}_j}{s - \dim G/2 + j} \right) \tag{2.5}$$

is continuous on $\mathscr{M}^{HL}(G) \times (D(\dim G/2 - N - 1) \setminus \operatorname{sing}(\dim G))$. For every $\mathbf{m} \in \mathscr{M}^{HL}(G)$, (2.5) *is meromorphic in $s \in D(\dim G/2 - N - 1)$. The set of poles is $\operatorname{sing}(\dim G) \cap D(\dim G/2 - N - 1)$. Every pole is of first order and is generated by exactly one term in the sum in* (2.5). *Furthermore, the s-derivative of* (2.5) *in $s = 0$ is continuous in $\mathbf{m} \in \mathscr{M}^{HL}(G)$ and is given by*

$$\mathbf{m} \mapsto \frac{\operatorname{Vol}(\Gamma\backslash G, \mathbf{m})}{(4\pi)^{\dim G/2}} \left(\int_0^1 \left(\sigma^{\mathbf{m}}_0(t) - T^N_{+0}[\sigma^{\mathbf{m}}_0](t) \right) t^{-\dim G/2-1} \mathrm{d}t + \sum_{\substack{j=0 \\ j \neq \dim G/2}}^N \frac{a^{\mathbf{m}}_j}{j - \frac{\dim G}{2}} + \begin{cases} \gamma \cdot a^{\mathbf{m}}_{\dim G/2} & \text{if } \dim G \text{ is even}\,, \\ 0 & \text{if } \dim G \text{ is odd}\,. \end{cases} \right), \tag{2.6}$$

where $\gamma = -\Gamma'(1) = \lim_{k\to\infty} (1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} - \ln k)$ is Euler's constant.

PROOF. Note that the factor $1/\Gamma(s)$ in (2.5) produces a zero at every $s \in -\mathbb{N}_0$. Hence, in case $\dim G$ is even and $N > \dim G/2 - 1$, $1/\Gamma(s)$ cancels any pole generated by the sum in (2.5) that is not in $\operatorname{sing}(\dim G) \cap D(\dim G/2 - N - 1)$.

We will first show that the integral is a continuous function of $(\mathbf{m}, s)$ and a holomorphic function on $s \in D(\dim G/2 - N - 1)$ for every $\mathbf{m} \in \mathscr{M}^{HL}(G)$. Let $K \subset \mathscr{M}^{HL}(G)$ be compact and $D \subset D(\dim G/2 - N - 1)$ be a compact disc. By Taylor's theorem with Langrangian remainder we have for all $t \in [0, 1]$:

$$\left|\sigma_0^{\mathbf{m}}(t) - T_{+0}^N[\sigma_0^{\mathbf{m}}](t)\right| < \tfrac{1}{(N+1)!}\left|(\sigma_0^{\mathbf{m}})^{(N+1)}(\tau_t)\right| \cdot t^{N+1},$$

for some $\tau_t \in (0, 1)$. By Lemma 1.88, $K \times [0, 1] \ni (\mathbf{m}, t) \mapsto (\sigma_0^{\mathbf{m}})^{(N+1)}(t)$ is continuous. Hence, with $C := \sup_{(\mathbf{m},t)\in K\times[0,1]} |(\sigma_0^{\mathbf{m}})^{(N+1)}|$ we have

$$\left|\sigma_0^{\mathbf{m}}(t) - T_{+0}^N[\sigma_0^{\mathbf{m}}](t)\right| \leqslant C \cdot t^{N+1}$$

for all $\mathbf{m} \in K$ and $t \in [0, 1]$. Since D is compact there exists $\delta > 0$ such that $\Re s > \dim G/2 - N - 1 + \delta$. It follows that

$$\left|(\sigma_0^{\mathbf{m}}(t) - T_{+0}^N[\sigma_0^{\mathbf{m}}](t)) \cdot t^{s-\dim G/2-1}\right| \leqslant C \cdot t^{N+1} \cdot t^{\Re s-\dim G/2-1} \leqslant C \cdot t^{-1+\delta}.$$

The right hand side is integrable on $(0, 1]$. It follows from Theorems 1.36 and 1.38 that

$$(\mathbf{m}, s) \mapsto \int_0^1 \left(\sigma_0^{\mathbf{m}}(t) - T_{+0}^N[\sigma_0^{\mathbf{m}}](t)\right) t^{s-\dim G/2-1}\mathrm{d}t$$

is continuous on $\mathscr{M}^{HL}(G) \times D(\dim G/2 - N - 1)$ and holomorphic on $s \in D(\dim G/2 - N - 1)$ for every $\mathbf{m} \in \mathscr{M}^{HL}(G)$.

It remains to prove (2.6). We have just shown that (2.5) is of the form

$$\frac{1}{\Gamma(s)}\left(f(s) + I(s)\right),$$

where I is holomorphic in $s = 0$ and

$$f(s) = \begin{cases} \frac{a_{\dim G/2}}{s} & \text{if } \dim G \text{ is even},\\ 0 & \text{otherwise}.\end{cases}$$

By **[OLBC10**, 5.7.1**]** we have $\frac{1}{\Gamma(s)} = \sum_{j=1}^{\infty} b_j s^j$ with $b_1 = 1$ and $b_2 = \gamma$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}\frac{1}{\Gamma(s)}\left(f(s) + I(s)\right) = I(0) + \gamma \cdot f(1).$$

$\square$

THEOREM 2.5. *Let G be a connected and simply connected nonsingualar 2-step nilpotent Lie group, $\Gamma \subset G$ a uniform subgroup and let ℓ be the dimension of the centre of G and $n := (\dim G - \ell)/2$. Assume there exists $\mathbf{m} \in \mathscr{M}^{HL}(G)$. Let $N \in \mathbb{N}_0 \cup \{-1\}$ and $T^N_{+0}[\sigma^{\mathbf{m}}_0](t) = \sum_{j=0}^N a^{\mathbf{m}}_j t^j$. Then for every $(\mathbf{m}, s) \in \mathscr{M}(G)^{HL} \times D(\dim G/2 - N - 1)$ we have*

$$\zeta\left(\left(\Gamma\backslash G, \mathbf{m}\right), s\right) = \zeta_B\left(\left(\Gamma\backslash G, \mathbf{m}\right), s\right) + \zeta_F\left(\left(\Gamma\backslash G, \mathbf{m}\right), s\right) \tag{2.7}$$

where

$$\zeta_B\left(\left(\Gamma\backslash G, \mathbf{m}\right), s\right) := \frac{1}{\Gamma(s)}\left(-\frac{1}{s} + \int_1^\infty \sum_{\lambda\in\Gamma^*_{\mathfrak{n}}\backslash\{0\}} e^{-4\pi^2\|\lambda\|^2_{\mathbf{m}_{\mathfrak{n}}} t} t^{s-1}\mathrm{d}t + \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi)^n}\int_1^\infty \sum_{X\in\Gamma_{\mathfrak{n}}\backslash\{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_{\mathfrak{n}}}}{4}t} t^{n-s-1}\mathrm{d}t\right) \tag{2.8}$$

and

$$\begin{aligned}\zeta_F\left(\left(\Gamma\backslash G, \mathbf{m}\right), s\right) := {}& \frac{1}{\Gamma(s)}\left(\frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi)^n}\int_1^\infty \sum_{\lambda\in\Gamma^*_{\mathfrak{z}}\backslash\{0\}} s^{\mathbf{m}}_{V,t}(\lambda) t^{s-n-1}\mathrm{d}t + \frac{\operatorname{Vol}\left(\Gamma\backslash G, \mathbf{m}\right)}{(4\pi)^{\dim G/2}}\int_0^1 \sum_{X\in\Gamma_{\mathfrak{z}}\backslash\{0\}} \sigma^{\mathbf{m}}_X(t)\, t^{s-\dim G/2-1}\mathrm{d}t\right.\\ &\left. + \frac{\operatorname{Vol}\left(\Gamma\backslash G, \mathbf{m}\right)}{(4\pi)^{\dim G/2}}\left(\int_0^1 \left(\sigma^{\mathbf{m}}_0(t) - T^N_{+0}[\sigma^{\mathbf{m}}_0](t)\right) t^{s-\dim G/2-1}\mathrm{d}t + \sum_{j=0}^N \frac{a^{\mathbf{m}}_j}{s - \frac{\dim G}{2} + j}\right)\right).\end{aligned} \tag{2.9}$$

The function

$$\mathscr{M}^{HL}(G) \times \mathbb{C} \ni (\mathbf{m}, s) \mapsto \zeta_B\left(\left(\Gamma\backslash G, \mathbf{m}\right), s\right)$$

is continuous and for every $\mathbf{m} \in \mathscr{M}(G)^{HL}$, $\mathbb{C} \ni s \mapsto \zeta_B((\Gamma\backslash G, \mathbf{m}), s)$ is entire. The function

$$\mathscr{M}^{HL}(G) \times (D(\dim G/2 - N - 1)\backslash \operatorname{sing}(\dim G)) \ni (\mathbf{m}, s) \mapsto \zeta_F\left(\left(\Gamma\backslash G, \mathbf{m}\right), s\right)$$

is continuous and for every $\mathbf{m} \in \mathscr{M}(G)^{HL}$, $D(\dim G/2 - N - 1) \ni s \mapsto \zeta_F\left(\left(\Gamma\backslash G, \mathbf{m}\right), s\right)$ is meromorphic with pole set $\operatorname{sing}(\dim G) \cap D(\dim G/2 - N - 1)$. Every pole is of first order and generated by exactly one term in the sum in the last line of (2.9).

PROOF. Let $\mathbf{m} \in \mathscr{M}^{HL}(G)$ and $s \in \mathbb{C}$ with $\Re s > \dim G/2$. We start with the defining Dirichlet series of the spectral ζ-function of $(\Gamma\backslash G, \mathbf{m})$:

$$\zeta((\Gamma\backslash G, \mathbf{m}), s) = \sum_{\substack{\sigma\in\Sigma(\Gamma\backslash G, \mathbf{m})\\ \sigma\neq 0}} \sigma^{-s}. \tag{2.10}$$

Let now $K(t)$ be the heat trace of $(\Gamma\backslash G, \mathbf{m})$. Then $K(t)$ and (2.10) are related by the Mellin transform (see, e.g., [**OLBC10**, 1.14(iv)]):

$$\zeta((\Gamma\backslash G, \mathbf{m}), s) = \sum_{\substack{\sigma\in\Sigma(\Gamma\backslash G,\mathbf{m})\\ \sigma\neq 0}} \sigma^{-s} = \frac{1}{\Gamma(s)} \sum_{\substack{\sigma\in\Sigma(\Gamma\backslash G,\mathbf{m})\\ \sigma\neq 0}} \int_0^\infty e^{-\sigma t} t^{s-1} \mathrm{d}t$$

$$(2.11) \qquad = \frac{1}{\Gamma(s)} \int_0^\infty \sum_{\substack{\sigma\in\Sigma(\Gamma\backslash G,\mathbf{m})\\ \sigma\neq 0}} e^{-\sigma t} t^{s-1} \mathrm{d}t = \frac{1}{\Gamma(s)} \int_0^\infty (K(t)-1) t^{s-1} \mathrm{d}t\,.$$

Integration and summation can be exchanged due to the uniform convergence of the the heat trace series. We split the integral $\int_0^\infty$ into $\int_0^1 + \int_1^\infty$ and use two different expressions for $K(t)$, the one given in (1.39), (1.40) and (1.41) and the other one given in Theorem 1.78*(i)*:

(2.12)

$$\begin{aligned}
\zeta((\Gamma\backslash G, \mathbf{m}), s) &= \frac{1}{\Gamma(s)} \int_0^1 \left(\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}} \sum_{X\in\Gamma_\mathfrak{n}\backslash\{0\}} \widehat{s^{\mathbf{m}}_{H,t}}(X) + \frac{\operatorname{Vol}(\Gamma\backslash G,\mathbf{m})}{(4\pi t)^n} \sum_{X\in\Gamma_\mathfrak{z}} \widehat{s^{\mathbf{m}}_{V,t}}(X) - 1 \right) t^{s-1}\mathrm{d}t \\
&\quad + \frac{1}{\Gamma(s)} \int_1^\infty \left(\sum_{\lambda\in\Gamma^*_\mathfrak{n}\backslash\{0\}} s^{\mathbf{m}}_{H,t}(\lambda) + \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi t)^n} \sum_{\lambda\in\Gamma^*_\mathfrak{z}\backslash\{0\}} s^{\mathbf{m}}_{V,t}(\lambda) \right) t^{s-1}\mathrm{d}t \\
&= \frac{1}{\Gamma(s)} \left(-\frac{1}{s} + \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n} \int_0^1 \sum_{X\in\Gamma_\mathfrak{n}\backslash\{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_\mathfrak{n}}}{4t}} t^{s-n-1}\mathrm{d}t + \int_1^\infty \sum_{\lambda\in\Gamma^*_\mathfrak{n}\backslash\{0\}} e^{-4\pi^2\|\lambda\|^2_{\mathbf{m}_\mathfrak{n}} t} t^{s-1}\mathrm{d}t \right) \\
&\quad + \frac{1}{\Gamma(s)} \left(\frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n} \int_1^\infty \sum_{\lambda\in\Gamma^*_\mathfrak{z}\backslash\{0\}} s^{\mathbf{m}}_{V,t}(\lambda) t^{s-n-1}\mathrm{d}t + \frac{\operatorname{Vol}(\Gamma\backslash G,\mathbf{m})}{(4\pi)^{\dim G/2}} \int_0^1 \sum_{X\in\Gamma_\mathfrak{z}\backslash\{0\}} \sigma^{\mathbf{m}}_X(t)\, t^{s-\dim G/2-1}\mathrm{d}t \right. \\
&\qquad\qquad \left. + \frac{\operatorname{Vol}(\Gamma\backslash G,\mathbf{m})}{(4\pi)^{\dim G/2}} \int_0^1 \sigma^{\mathbf{m}}_0(t)\cdot t^{s-\dim G/2-1}\mathrm{d}t \right).
\end{aligned}$$

The third line of (2.12) is, up to a transformation of the first integral by the diffeomorphism $t \mapsto 1/t$, the expression for $\zeta_B((\Gamma\backslash G), s)$. The factor $1/\Gamma(s)$ is entire and its zero at $s = 0$ cancels the pole of $1/s$. The two integrals are continuous in $(\mathbf{m}, s)$ and entire in $s \in \mathbb{C}$ by Proposition 1.93 and Lemmata 2.2, 2.3.

The fourth line of (2.12) is continuous in $(\mathbf{m}, s)$ and entire in $s \in \mathbb{C}$ by Propositions 1.82, 1.93 and Lemmata 2.2 and 2.3. The fifth line is continuous in $(\mathbf{m}, s)$ and holomorphic in $s \in D(\dim G/2)$ by Lemma 2.4 (choose $N = -1$ there). We obtain its meromorphic continuation to $D(\dim G/2 - N - 1)$ by a simple trick:

$$\frac{1}{\Gamma(s)}\int_0^1 \sigma_0^{\mathbf{m}}(t)\cdot t^{s-\dim G/2-1}\mathrm{d}t = \frac{1}{\Gamma(s)}\int_0^1 \Big(\sigma_0^{\mathbf{m}}(t) - T^N_{+0}[\sigma_0^{\mathbf{m}}](t) + T^N_{+0}[\sigma_0^{\mathbf{m}}](t)\Big)\cdot t^{s-\dim G/2-1}\mathrm{d}t$$

$$= \frac{1}{\Gamma(s)}\left(\int_0^1 \Big(\sigma_0^{\mathbf{m}}(t) - T^N_{+0}[\sigma_0^{\mathbf{m}}](t)\Big)\cdot t^{s-\dim G/2-1}\mathrm{d}t + \sum_{j=0}^{N}\frac{a_j^{\mathbf{m}}}{s-\dim G/2+j}\right)$$

The remaining statements about $\zeta_F((\Gamma\backslash G,\mathbf{m}),s)$ now follow from Lemma 2.4. □

COROLLARY 2.6. *Let G be a connected and simply connected nonsingualar 2-step nilpotent Lie group of dimension $\ell+2n$, where ℓ is the dimension of the centre of G. Let $\Gamma\subset G$ be a uniform subgroup. Assume there exists $\mathbf{m}\in\mathscr{M}^{HL}(G)$. Let $N\in\mathbb{N}$ with $N\geqslant\lfloor\frac{\dim G}{2}\rfloor$ and $T^N_{+0}[\sigma_0^{\mathbf{m}}](t)=\sum_{j=0}^N a_j^{\mathbf{m}}t^j$. Then for every $\mathbf{m}\in\mathscr{M}(G)^{HL}$ the height of $(\Gamma\backslash G,\mathbf{m})$ is given by*

$$\zeta'\left((\Gamma\backslash G,\mathbf{m}),0\right) = \zeta_B'\left((\Gamma\backslash G,\mathbf{m}),0\right) + \zeta_F'\left((\Gamma\backslash G,\mathbf{m}),0\right) \tag{2.13}$$

where

$$\zeta_B'\left((\Gamma\backslash G,\mathbf{m}),0\right) = -\gamma + \int_1^\infty \sum_{\lambda\in\Gamma_{\mathfrak{n}}^*\backslash\{0\}} e^{-4\pi^2\|\lambda\|^2_{\mathbf{m}_{\mathfrak{n}}}t}t^{-1}\mathrm{d}t + \frac{\operatorname{Vol}T_{\mathfrak{n},\mathbf{m}}}{(4\pi)^n}\int_1^\infty \sum_{X\in\Gamma_{\mathfrak{n}}\backslash\{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_{\mathfrak{n}}}}{4}t}t^{n-1}\mathrm{d}t \tag{2.14}$$

and

$$\begin{aligned}\zeta_F'\left((\Gamma\backslash G,\mathbf{m}),0\right) = {}& \frac{\operatorname{Vol}T_{\mathfrak{n},\mathbf{m}}}{(4\pi)^n}\int_1^\infty \sum_{\lambda\in\Gamma_{\mathfrak{z}}^*\backslash\{0\}} s^{\mathbf{m}}_{V,t}(\lambda)t^{-n-1}\mathrm{d}t + \frac{\operatorname{Vol}(\Gamma\backslash G,\mathbf{m})}{(4\pi)^{\dim G/2}}\int_0^1 \sum_{X\in\Gamma_{\mathfrak{z}}\backslash\{0\}}\sigma_X^{\mathbf{m}}(t)\,t^{-\dim G/2-1}\mathrm{d}t \\ &+ \frac{\operatorname{Vol}(\Gamma\backslash G,\mathbf{m})}{(4\pi)^{\dim G/2}}\left(\int_0^1\Big(\sigma_0^{\mathbf{m}}(t) - T^N_{+0}[\sigma_0^{\mathbf{m}}](t)\Big)t^{-\dim G/2-1}\mathrm{d}t + \sum_{\substack{j=0\\ j\neq\dim G/2}}^{N}\frac{a_j^{\mathbf{m}}}{j-\dim G/2}\right. \\ &\left. + \begin{cases}\gamma\cdot a^{\mathbf{m}}_{\dim G/2} & \text{if } \dim G \text{ is even},\\ 0 & \text{if } \dim G \text{ is odd}.\end{cases}\right).\end{aligned} \tag{2.15}$$

Here, γ denotes Euler's constant. Moreover, the functions $\mathscr{M}(G)^{HL}\ni\mathbf{m}\mapsto\zeta_B'((\Gamma\backslash G,\mathbf{m}),0)$ and $\mathscr{M}(G)^{HL}\ni\mathbf{m}\mapsto\zeta_B'((\Gamma\backslash G,\mathbf{m}),0)$ are continuous.

PROOF. By the assumption on N we have $0\in D(\dim G/2-N-1)$. Hence we can use formulas (2.7), (2.8) and (2.9). In the course of the proofs of Lemma 2.4 and Theorem 2.5

we have shown that these expressions are of the form

$$\zeta\left((\Gamma\backslash G,\mathbf{m}),s\right)=\frac{1}{\Gamma(s)}\left(-\frac{1}{s}+f(s)+I(s)\right),$$

where I is holomorphic in $s=0$ and

$$f(s)=\begin{cases}\frac{a_{\dim G/2}}{s} & \text{if } \dim G \text{ is even}\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Since $\frac{1}{\Gamma(s)}=\sum_{j=1}^{\infty}b_js^j$ with $b_1=1$ and $b_2=\gamma$ (see [**OLBC10**, 5.7.1]) we have

$$\zeta'\left((\Gamma\backslash G,\mathbf{m}),0\right)=\gamma(f(1)-1)+I(0)\,.$$

This proves formulas (2.13), (2.14) and (2.15). The continuity of $\mathbf{m}\mapsto\zeta_B'((\Gamma\backslash G,\mathbf{m}),0)$ and $\mathbf{m}\mapsto\zeta_F'((\Gamma\backslash G,\mathbf{m}),0)$ now follows from Proposition 1.93 and Lemmata 2.2, 2.3, 2.4. ☐

REMARK 2.7. The ζ-function $\zeta((\Gamma\backslash G,\mathbf{m}),s)$ and the height $\zeta'((\Gamma\backslash G,\mathbf{m}),0)$ of a nonsingular Heisenberg-like nilmanifold are spectral objects. As such, they only depend on the isometry class of $(\Gamma\backslash G,\mathbf{m})$. Therefore, we obtain well-defined and continuous functions $\mathscr{M}^{HL}(\Gamma,G)\ni\mathbf{m}\mapsto\zeta((\Gamma\backslash G,\mathbf{m}),s)$ and $\mathscr{M}^{HL}(\Gamma,G)\ni\mathbf{m}\mapsto\zeta'((\Gamma\backslash G,\mathbf{m}),0)$, where $\mathscr{M}^{HL}(\Gamma,G)$ is the moduli space of Heisenberg-like metrics on $\Gamma\backslash G$ from Definition 1.43.

REMARK 2.8. Building on Remark 1.79 we observe that $\zeta_B((\Gamma\backslash G,\mathbf{m}),s)$ and $\zeta_F((\Gamma\backslash G,\mathbf{m}),s)$ (resp. ζ_B' and ζ_F') depend on different parts of the metric $\mathbf{m}$. Namely, $\zeta_B((\Gamma\backslash G,\mathbf{m}),s)$ depends only on $\mathbf{m}_{\mathfrak{n}}$ and $\zeta_F((\Gamma\backslash G,\mathbf{m}),s)$ depends only on $\mathbf{m}_{\mathfrak{z}}$, $c_1^{\mathbf{m}},\dots,c_n^{\mathbf{m}}$ and the total volume $\operatorname{Vol}(\Gamma\backslash G,\mathbf{m})=\operatorname{Vol}T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}\cdot\operatorname{Vol}T_{\mathfrak{z},\mathbf{m}_{\mathfrak{z}}}$.

In case $(\Gamma\backslash G,\mathbf{m})$ is a normalised Heisenberg manifold $(\Gamma^r\backslash H_n)$ with $\mathbf{m}=(h,g)$ as in (1.28) we have $\mathbf{m}_{\mathfrak{n}}=h$, $\mathbf{m}_{\mathfrak{z}}=g$, $c_j^{\mathbf{m}}=g^{1/2}d_j(h)$ (see Proposition 1.49) and $\operatorname{Vol}(\Gamma^r\backslash H_n,(h,g))=|\Gamma^r|\sqrt{g\det h}=|\Gamma^r|\sqrt{g}d_1(h)^{-1}\cdots d_n(h)^{-1}$ (see Remark 1.71). Hence, $\zeta_B((\Gamma^r\backslash H_n,(h,g)),s)$ only depends on h whereas $\zeta_F((\Gamma^r\backslash H_n,(h,g)),s)$ only depends on g and $d_1(h),\dots,d_n(h)$. By Proposition 1.48 this means that

$$\mathcal{P}_{2n}^*(h_0)\ni h\mapsto\zeta_F((\Gamma^r\backslash H_n,(h,g)),s)$$

is constant for any $h_0\in\mathcal{P}_{2n}$ and fixed $g\in(0,\infty)$. A situation of this kind arises in the context of volume normalised Heisenberg manifolds of Heisenberg-type. Recall from Remark 1.71:

$$\mathcal{SM}_n^{r,HT}=\left\{[(h,g)]\in\mathcal{M}_n^r\,|\,g=|\Gamma^r|^{-\frac{2}{n+1}}\text{ and }h\in\mathcal{P}_{2n}^*(g^{1/2}\cdot\mathrm{Id})\right\}.$$

It follows that

$$\mathcal{SM}_n^{r,HT}\ni\mathbf{m}\mapsto\zeta_F((\Gamma^r\backslash H_n),\mathbf{m})$$

is constant.

2. (Non-)Existence of Global Minima

This section explores conditions under wich $[\mathbf{m}] \mapsto \zeta'((\Gamma\backslash G, \mathbf{m}), 0)$ and $[\mathbf{m}] \mapsto \zeta((\Gamma\backslash G, \mathbf{m}), s)$ with $s \in (0, \infty)$ not a pole attain global minima.

First, we consider Heisenberg type metrics. Theorem 2.10 states that both functions are bounded from below on the moduli space of volume normalised Heisenberg type metrics. Moreover, in case the centre of G is 1-dimensional, both functions attain global minima.

Next, we consider Heisenberg-like metrics. Theorem 2.11*(i)* establishes boundedness from below of the height in case $\dim \Gamma\backslash G \equiv 3 \pmod 4$. As for the ζ-function, we restrict to the interval $(0, {}^{\dim G}/2)$. The poles of $\zeta((\Gamma\backslash G), s)$, located at $\operatorname{sing}(\dim G) = \{{}^{\dim G}/2, {}^{\dim G}/2 - 1, \ldots\}$, divide the interval $(0, {}^{\dim G}/2)$ into $\lfloor {}^{\dim G}/2 \rfloor + 1$ subintervalls. Theorem 2.11*(ii)* establishes boundedness of the ζ-function in case s is contained in every other of these subintervalls. For G with 1-dimensional centre, both of these statements are then strengthened in Theorem 2.11*(iii)* and *(iv)*, respectively. Under the same assumptions, the height and the ζ-function attain global minima.

The inevitable question then is: what happens to the height in dimensions $\not\equiv 3 \pmod 4$ and to the ζ-function for s not covered by Theorem 2.11*(ii)*?

In Proposition 2.12 we introduce for any Heisenberg-like metric $\mathbf{m}$ a path $\mathbf{m}_\rho$ of Heisenberg-like metrics with $\mathbf{m}_1 = \mathbf{m}$. If $\rho \to \infty$ then $[\mathbf{m}_\rho]$ approaches the boundary at infinity of the moduli space of Heisenberg-like metrics. Theorem 2.19*(i)* establishes that the height is not bounded from below along $\mathbf{m}_\rho$ in case $\dim(\Gamma\backslash G) \equiv 0 \pmod 4$ or $\dim(\Gamma\backslash G) \equiv 1 \pmod 4$. Accordingly, Theorem 2.19*(ii)* states that the ζ-function is not bounded from below along $\mathbf{m}_\rho$ if s is in a subintervall of $(0, {}^{\dim G}/2)\backslash \operatorname{sing}(\dim G)$ not covered by Theorem 2.11*(ii)*.

The path $\mathbf{m}_\rho$ represents the most basic deformation of a metric we can think of in the context of nilmanifolds; along it, we simply scale the metrics of the base torus $T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}$ and the fibre torus $T_{\mathfrak{z},\mathbf{m}_\mathfrak{z}}$ in such a way that the total volume of $(\Gamma\backslash G, \mathbf{m}_\rho)$ remains constant. Unfortunately, it is also the only constant volume metric deformation of a Heisenberg-like metric through Heisenberg-like metrics that exists in general. We therefore change the setting to Heisenberg manifolds.

Definition 2.20 provides for any Heisenberg manifold $(\Gamma^r\backslash H_n)$ a path $f_{n,r}(\rho)$ of metrics. This path also approaches the boundary at infinity of the moduli space of volume normalised Heisenberg-like metrics, but the volumes of the base and fibre tori stay constant along it. Nevertheless, Theorem 2.28 establishes facts for the path $f_{n,r}(\rho)$ analogous to those in Theorem 2.19: if $\dim(\Gamma^r\backslash H_n) \equiv 1 \pmod 4$ then the height of $(\Gamma^r\backslash H_n, f_{n,r}(\rho))$

is not bounded from below as $\rho \to \infty$. Similarly, for s in positions as in Theorem 2.19*(ii)*, the ζ-function of $(\Gamma^r\backslash H_n, f_{n,r}(\rho))$ is not bounded from below as $\rho \to \infty$.

What properties do the paths $\mathbf{m}_\rho$ and $f_{n,r}(\rho)$ share? One chan show that the sectional curvature w.r.t. each of these paths grows unboundedly as $\rho \to \infty$.

Thus, Theorem 2.30 concludes all of the above considerations by establishing the boundedness from below of the height and the ζ-function on the moduli space of volume normalised metrics whose sectional curvature is uniformly bounded from above. As in Theorem 2.11, both of these functions even attain global minima if the dimension of the centre of G is 1. With respect to the height, the author does not know whether the additional assumption on the sectional curvature is necessary in case $\dim(\Gamma\backslash G) \equiv 2 \pmod 4$.

At last in this section, we reprove results from **[Fd03]**. There, K. Furutani and S. de Gosson presented formulas for the height of two discrete series of respectively 3- and 5-dimensional Heisenberg type nilmanifolds and proved that the height grows unboundedly as the series parameter goes to infinity. While we cannot reproduce their formula for the height using ours, we can reprove their asymptotic result which we do in Proposition 2.32. In fact, the methods we develop allow us to generalise their result to arbitrary dimensions.

Patrick Chiu proved in [**Chi97**, Theorem 3.1] that the height attains a minimum on the moduli space of flat tori with volume one. The following lemma captures a variation of his argument and will be used in the proofs of Theorem 2.10, Theorem 2.11 and Theorem 2.30.

Recall from Notation and Remarks 1.51(ii) the definition of the first minimum $m(Y)$ of a positive definite symmetric matrix $Y \in \mathcal{P}_n$:

$$m(Y) = \inf\{Y[a] \mid a \in \mathbb{Z}^n \backslash \{0\}\}\,.$$

LEMMA 2.9. *Let $n \in \mathbb{N}$, $C > 0$, $\tau \in \mathbb{R}$ and*

$$F : \mathcal{P}_n \ni Y \mapsto \int_1^\infty \sum_{X \in \mathbb{Z}^n \backslash \{0\}} e^{-CY[X]t} t^{\tau - 1} \mathrm{d}t \in \mathbb{R}\,,$$

which is well-defined by Remark 1.92 and Lemma 2.3. Then we have

$$F(Y) \to \infty \quad \text{as} \quad m(Y) \to 0\,.$$

PROOF. Fix an arbitrary $T > 1$. Since all terms of the integrand are positive functions, we have

$$\int_1^\infty \sum_{X \in \mathbb{Z}^n \setminus \{0\}} e^{-CY[X]t} t^{\tau-1} \, \mathrm{d}t \geqslant \int_1^T \sum_{X \in \mathbb{Z}^n \setminus \{0\}} e^{-CY[X]t} t^{\tau-1} \, \mathrm{d}t \,. \tag{2.16}$$

We discard all terms but those which are indexed by an integral multiple of a fixed shortest vector of the lattice with Gram matrix Y. Thus, we get the following lower estimate for the right hand side of (2.16):

$$\begin{aligned}
\int_1^T 2 \sum_{n=1}^\infty e^{-Cm(Y)n^2 t} t^{\tau-1} \, \mathrm{d}t &\geqslant \int_1^T 2 \int_1^\infty e^{-Cm(Y)x^2 t} \, \mathrm{d}x t^{\tau-1} \, \mathrm{d}t \\
&= \int_1^T 2 \int_{\sqrt{Cm(Y)t}}^\infty e^{-u^2} \frac{1}{\sqrt{Cm(Y)t}} \, \mathrm{d}u \, t^{\tau-1} \, \mathrm{d}t \\
&= \sqrt{\frac{\pi}{Cm(Y)}} \int_1^T \operatorname{erfc}\left(\sqrt{Cm(Y)t}\right) t^{\tau-3/2} \, \mathrm{d}t \,,
\end{aligned}$$

where $\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-u^2} \, \mathrm{d}u$ is the complementary error function (see, e.g. [**OLBC10**, Ch. 7]). Viewed as a function on the nonnegative real axis, $0 < \operatorname{erfc} \leqslant 1$ is a monotonously decreasing function with $\operatorname{erfc}(0) = 1$. We thus have

$$\begin{aligned}
\sqrt{\frac{\pi}{Cm(Y)}} \int_1^T \operatorname{erfc}\left(\sqrt{Cm(Y)t}\right) t^{\tau-3/2} \, \mathrm{d}t &\geqslant \frac{\sqrt{\pi} \operatorname{erfc}\left(\sqrt{Cm(Y)T}\right)}{\sqrt{Cm(Y)}} \int_1^T t^{\tau-3/2} \, \mathrm{d}t \\
&\to \infty \quad \text{as} \quad m(Y) \to 0 \,.
\end{aligned}$$

□

Now recall from Definition 1.70 the moduli space $\mathscr{SM}^{HT}(\Gamma, G)$ and from Remark 1.69*(ii)* the definition of the boundary (at infinity) $\partial\mathscr{SM}^{HT}(\Gamma, G)$.

THEOREM 2.10. *Let G be a connected and simply connected nonsingular 2-step nilpotent Lie group, $\Gamma \subset G$ a uniform subgroup and $s \in (0, \infty) \setminus \operatorname{sing}(\dim G)$. Then, the ζ-function $[\mathbf{m}] \mapsto \zeta((\Gamma\backslash G, \mathbf{m}), s)$ and the height $[\mathbf{m}] \mapsto \zeta'((\Gamma\backslash G, \mathbf{m}), 0)$ are bounded from below on the moduli space $\mathscr{SM}^{HT}(\Gamma, G)$ of isometry classes of metrics of Heisenberg type and with volume one on $\Gamma\backslash G$.*

Moreover, if the dimension of the centre of G is one then this statement can be strengthened as follows: The height $[\mathbf{m}] \mapsto \zeta'((\Gamma\backslash G, \mathbf{m}), 0)$ and the ζ-function $[\mathbf{m}] \mapsto \zeta((\Gamma\backslash G, \mathbf{m}), s)$ both *go to* $+\infty$ *as* $[\mathbf{m}] \to \partial\mathscr{S}\mathscr{M}^{HT}(\Gamma, G)$. *Consequently, both functions attain a global minimum on* $\mathscr{S}\mathscr{M}^{HT}(\Gamma, G)$.

PROOF. We will work with the function $[\mathbf{m}] \mapsto \Gamma(s)\zeta((\Gamma\backslash G, \mathbf{m}), s)$ instead of $[\mathbf{m}] \mapsto \zeta((\Gamma\backslash G, \mathbf{m}), s)$. The result is the same since $\Gamma(s) > 0$ for all $s \in (0, \infty)$.

By Theorem 2.5 and Corollary 2.6 the functions $[\mathbf{m}] \mapsto \Gamma(s)\zeta((\Gamma\backslash G, \mathbf{m}), s)$ and $[\mathbf{m}] \mapsto \zeta'((\Gamma\backslash G, \mathbf{m}), 0)$ are of the form

$$
\begin{aligned}
(2.17)\quad & C(\tau) + \int_1^\infty \sum_{\lambda \in \Gamma_{\mathfrak{n}}^* \backslash \{0\}} e^{-4\pi^2 \|\lambda\|^2_{\mathbf{m}_{\mathfrak{n}}} t} t^{\tau - 1} \mathrm{d}t + \frac{\operatorname{Vol} T_{\mathfrak{n}, \mathbf{m}_{\mathfrak{n}}}}{(4\pi)^n} \int_1^\infty \sum_{X \in \Gamma_{\mathfrak{n}} \backslash \{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_{\mathfrak{n}}}}{4} t} t^{n - \tau - 1} \mathrm{d}t \\
& + \frac{\operatorname{Vol} T_{\mathfrak{n}, \mathbf{m}_{\mathfrak{n}}}}{(4\pi)^n} \int_1^\infty \sum_{\lambda \in \Gamma_{\mathfrak{z}}^* \backslash \{0\}} s^{\mathbf{m}}_{V,t}(\lambda) t^{\tau - n - 1} \mathrm{d}t + \frac{1}{(4\pi)^{\dim G/2}} \int_0^1 \sum_{X \in \Gamma_{\mathfrak{z}} \backslash \{0\}} \sigma^{\mathbf{m}}_X(t) t^{\tau - \dim G/2 - 1} \mathrm{d}t \\
& + \frac{1}{(4\pi)^{\dim G/2}} \left(\int_0^1 \left(\sigma^{\mathbf{m}}_0(t) - T^N_{+0}[\sigma^{\mathbf{m}}_0](t) \right) t^{\tau - \dim G/2 - 1} \mathrm{d}t + \sum_{\substack{j=0 \\ j \neq \dim G/2}}^{N} \frac{a^{\mathbf{m}}_j}{\tau - \dim G/2 + j} \right. \\
& \qquad \left. + \begin{cases} \gamma \cdot a^{\mathbf{m}}_{\dim G/2} & \text{if } \dim G \text{ is even and } \tau = 0, \\ 0 & \text{if } \dim G \text{ is odd}, \\ \frac{a^{\mathbf{m}}_{\dim G/2}}{s} & \text{if } \dim G \text{ is even and } \tau = s \end{cases} \right),
\end{aligned}
$$

where τ equals s or 0, respectively, $C(\tau) \in \mathbb{R}$ is a constant, $N = \lfloor \frac{\dim G}{2} \rfloor$ or $N \in \mathbb{N}_0 \cup \{-1\}$ such that $s = \Re s > \dim G/2 - N - 1$ and $T^N_{+0}[\sigma^{\mathbf{m}}_0](t) = \sum_{j=0}^N a^{\mathbf{m}}_j t^j$.

The first line of (2.17) is clearly bounded from below. The first term in the second line is positive by definition of $s^{\mathbf{m}}_{V,t}$ (see (1.42)), the second term is positive by Corollary 1.84. The formula for $\sigma^{\mathbf{m}}_0$ given in (1.47) shows that $\sigma^{\mathbf{m}}_0$ also only depends on $c^{\mathbf{m}}_1, \ldots, c^{\mathbf{m}}_n$, but not on any other part of the metric $\mathbf{m}$. Since the $a^{\mathbf{m}}_j$ are Taylor coefficients of $\sigma^{\mathbf{m}}_0$, the third and fourth lines of (2.17) only depend on $c^{\mathbf{m}}_1, \ldots, c^{\mathbf{m}}_n$. But by the very definition of a Heisenberg type metric, $c^{\mathbf{m}}_1 = \cdots = c^{\mathbf{m}}_n = 1$ for every $[\mathbf{m}] \in \mathscr{S}\mathscr{M}^{HT}(\Gamma, G)$. It follows that (2.17) is bounded from below on $\mathscr{S}\mathscr{M}^{HT}(\Gamma, G)$.

Let now the centre of G be one dimensional. W.l.o.g. we can assume that $G = H_n$, the $(2n+1)$ dimensional Heisenberg group, and $\Gamma = \Gamma^r$ (see Theorem 1.44). We work with the homeomorphic moduli space $\mathcal{SM}^{r,HT}_n \simeq \mathscr{S}\mathscr{M}^{HT}(\Gamma^r, H_n)$ (see Definition 1.70). By Corollary 1.65, Remark 1.66 and Remark 1.71, the boundary $\partial\mathcal{SM}^{r,HT}_n$ is defined by $m_r(h) = 0$, where m_r was defined in Notation and Remarks 1.51(iii) and $[\mathbf{m}] = [(h, g)] \in \mathcal{SM}^{r,HT}_n$. This means that we have to prove that (2.17) goes to $+\infty$ as $m_r(h) \to 0$. Since we have shown that (2.17) is bounded from below, it suffices to show that the last term in the first line of (2.17) goes to $+\infty$ as $m_r(h) \to 0$. By Proposition 1.49 we have $\Gamma_{\mathfrak{n}}^r = \delta_r \cdot \mathbb{Z}^{2n}$. Hence, $\{\|X\|^2_{\mathbf{m}_{\mathfrak{n}}} \mid X \in \Gamma_{\mathfrak{n}}^r\} = \{h[\delta_r][X] \mid X \in \mathbb{Z}^{2n}\}$. The last term in the first line of (2.17) can therefore be written as

$$\frac{\operatorname{Vol} T_{\mathfrak{n},h}}{(4\pi)^n} \int_1^\infty \sum_{X \in \mathbb{Z}^{2n}\setminus\{0\}} e^{-\frac{h[\delta_r][X]}{4}t} t^{n-\tau-1} \mathrm{d}t\,.$$

Recall from Remark 1.71 that $[\mathbf{m}] = [(h,g)] \mapsto \det h$ is constant on $\mathcal{SM}_n^{r,HT}$. Hence, $\operatorname{Vol} T_{\mathfrak{n},h} = |\Gamma^r| \det h^{1/2}$ is constant. The result now follows from $m_r(h) = m(h[\delta_r])$ and Lemma 2.9. ☐

THEOREM 2.11. *Let G be a connected and simply connected nonsingular 2-step nilpotent Lie group, $\Gamma \subset G$ a uniform subgroup and $s \in (0, \dim G/2)\setminus \operatorname{sing}(\dim G)$. Then we have:*

- *(i) If $\dim G$ and $\lfloor \dim G/2 \rfloor$ are odd, i.e., $\dim G \equiv 3 \pmod 4$, then $[\mathbf{m}] \mapsto \zeta'((\Gamma\backslash G, \mathbf{m}), 0)$ is bounded from below on the moduli space $\mathscr{SM}^{HL}(\Gamma, G)$ of isometry classes of volume normalised Heisenberg-like metrics on $\Gamma\backslash G$.*
- *(ii) Assume $s \in (\dim G/2 - \mu - 1, \dim G/2 - \mu)$ with $\mu \in [0, \dim G/2) \cap \mathbb{N}_0$. If μ is odd, then $[\mathbf{m}] \mapsto \zeta((\Gamma\backslash G, \mathbf{m}), s)$ is bounded from below on $\mathscr{SM}^{HL}(\Gamma, G)$.*

Moreover, if the centre of G is one dimensional, then (i) and (ii) can be replaced by:

- *(iii) If $\dim G \equiv 3 \pmod 4$ then $[\mathbf{m}] \mapsto \zeta'((\Gamma\backslash G, \mathbf{m}), 0)$ attains a global minimum on $\mathscr{SM}^{HL}(\Gamma, G)$.*
- *(iv) Assume $s \in (\dim G/2 - \mu - 1, \dim G/2 - \mu)$ with $\mu \in [0, \dim G/2) \cap \mathbb{N}_0$. If μ is odd, then $[\mathbf{m}] \mapsto \zeta((\Gamma\backslash G, \mathbf{m}), s)$ attains a global minimum on $\mathscr{SM}^{HL}(\Gamma, G)$.*

PROOF. We prove *(i)* and *(ii)* together. Choose $N := \lfloor \dim G/2 \rfloor$ in *(i)* and $N := \mu$ in *(ii)*. By Theorem 2.5 and Corollary 2.6 the functions $[\mathbf{m}] \mapsto \Gamma(s)\zeta((\Gamma\backslash G, \mathbf{m}), s)$ and $[\mathbf{m}] \mapsto \zeta'((\Gamma\backslash G, \mathbf{m}), 0)$ are of the form

$$\begin{aligned}(2.18)\quad C(\tau) &+ \int_1^\infty \sum_{\lambda \in \Gamma_{\mathfrak{n}}^*\setminus\{0\}} e^{-4\pi^2\|\lambda\|^2_{\mathbf{m}_{\mathfrak{n}}} t} t^{\tau-1}\mathrm{d}t + \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi)^n} \int_1^\infty \sum_{X \in \Gamma_{\mathfrak{n}}\setminus\{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_{\mathfrak{n}}}}{4}t} t^{n-\tau-1}\mathrm{d}t \\ &+ \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}}{(4\pi)^n} \int_1^\infty \sum_{\lambda \in \Gamma_{\mathfrak{z}}^*\setminus\{0\}} s^{\mathbf{m}}_{V,t}(\lambda) t^{\tau-n-1}\mathrm{d}t + \frac{1}{(4\pi)^{\dim G/2}} \int_0^1 \sum_{X \in \Gamma_{\mathfrak{z}}\setminus\{0\}} \sigma_X^{\mathbf{m}}(t) t^{\tau-\dim G/2-1}\mathrm{d}t \\ &+ \frac{1}{(4\pi)^{\dim G/2}} \left(\int_0^1 \left(\sigma_0^{\mathbf{m}}(t) - T^N_{+0}[\sigma_0^{\mathbf{m}}](t)\right) t^{\tau-\dim G/2-1}\mathrm{d}t + \sum_{j=0}^N \frac{a_j^{\mathbf{m}}}{\tau - \dim G/2 + j} \right),\end{aligned}$$

where τ equals s or 0 respectively, $C(\tau) \in \mathbb{R}$ is a constant, and $T^N_{+0}[\sigma_0^{\mathbf{m}}](t) = \sum_{j=0}^N a_j^{\mathbf{m}} t^j$.

Note the absence of the term in line 4 of (2.15). This is because we assume $\dim G$ to be odd in *(i)*.

As functions of the metric $\mathbf{m}$, the first line of (2.18) is clearly bounded from below. The first term in the second line is positive by definition of $s^{\mathbf{m}}_{V,t}$ (see (1.42)) and the second term is positive by Corollary 1.84. Now note that N is odd by definition. Hence, the integral on the third line of (2.18) is positive by Corollary 1.89. Next, note that by Lemma 1.90 the coefficients $a^{\mathbf{m}}_j$ are homogeneous polynomials of order $2j$ in $c^{\mathbf{m}}_1, \ldots, c^{\mathbf{m}}_n$ and that the sign of the coefficient of every monomial in $a^{\mathbf{m}}_j$ is $(-1)^j$. Thus, the sum in the third line of (2.18) is a polynomial in $c^{\mathbf{m}}_1, \ldots, c^{\mathbf{m}}_n$. The highest degree monomials are all contained in $a^{\mathbf{m}}_N$. Their sign is $+1$ by definition of τ and since N is odd. It follows that (2.18) is bounded from below on $\mathscr{S}\mathscr{M}^{HL}(\Gamma, G)$.

We will now prove *(iii)* and *(iv)*. W.l.o.g. we assume $G = H_n$ and $\Gamma = \Gamma^r$ for some $r \in \mathcal{D}_n$ (see Theorem 1.44). Furthermore, we will work on the moduli space $\mathcal{SM}^r_n \simeq \mathscr{S}\mathscr{M}^{HL}(\Gamma^r, H_n)$ of normalised metrics on $\Gamma^r \backslash H_n$ (see Definition 1.70). The argument goes as follows. On every compact set of $\mathcal{SM}^r_n$, (2.18) will attain a minimum since the height and the ζ-function are continous on $\mathcal{SM}^r_n$. It thus suffices to show that (2.18) goes to $+\infty$ as $[\mathbf{m}] = [(h, g)] \to \partial\mathcal{SM}^r_n$.

The latter is by Corollary 1.64 the case if and only if one of the following cases is met (Recall that $g = |\Gamma^r|^{-2} \det h^{-1}$): $g \to 0$; $g \to \infty$; $m_r(h) \to 0$; $d_n(h) \to \infty$. This a priori constitutes 11 individual cases, depending on which of the functions g, m_r and d_n stay bounded or degenerate (e.g., $g \to 0$, m_r bounded from below and d_n bounded from above). We first show that we actually only need to consider 3 derived cases, which are: $m\left(h^{-1}\left[\delta_r^{-1}\right]\right) \to 0$; $g < b < \infty$ and $m_r\left(h\right) \to 0$; $g > a > 0$ and $d_n(h) \to \infty$.

If $g = |\Gamma^r|^{-2} \det h^{-1} \to 0$ then also $\det\left(h^{-1}\left[\delta_r^{-1}\right]\right) \to 0$. Hence by Remark 1.55 $m\left(h^{-1}\left[\delta_r^{-1}\right]\right) \to 0$, which is covered by the first case. On the other hand, if $g = |\Gamma^r|^{-2} \det h^{-1} \to \infty$ then $\det\left(h\left[\delta_r\right]\right) \to 0$ and again by Remark 1.55 $m_r(h) \to 0$. Recall that $\pm i d_1(h), \ldots, \pm i d_n(h)$ are the eigenvalues of $h^{-1}J$ (Definition 1.47). Hence, $g = |\Gamma^r|^{-2} \det h^{-1} = |\Gamma^r|^{-2}(d_1(h) \cdots d_n(h))^2$ and $g \to \infty$ implies $d_n(h) \to \infty$. We can thus w.l.o.g. assume that $g < b < \infty$ and $m_r\left(h\right) \to 0$ or $g > a \geqslant 0$ and $d_n(h) \to \infty$.

Now we look at the growth of (2.18) in each of the three cases.

- The first case we treat is $m\left(h^{-1}\left[\delta_r^{-1}\right]\right) \to 0$. By Proposition 1.49 we have $\Gamma^r_{\mathfrak{n}} = \delta_r \cdot \mathbb{Z}^{2n}$. Hence $\left\{4\pi^2\|\lambda\|^2_{\mathbf{m}_{\mathfrak{n}}} \mid \lambda \in (\Gamma^r_{\mathfrak{n}})^*\right\} = \left\{4\pi^2 h^{-1}[\delta_r^{-1}][X] \mid X \in \mathbb{Z}^{2n}\right\}$ and the first nonconstant term in (2.18) can be written as

$$\int_1^\infty \sum_{X \in \mathbb{Z}^{2n} \backslash \{0\}} e^{-4\pi^2 h^{-1}[\delta_r^{-1}][X]t} t^{\tau-1} \mathrm{d}t\,.$$

It now follows from Lemma 2.9 that (2.18) goes to $+\infty$ if $m\left(h^{-1}\left[\delta_r^{-1}\right]\right) \to 0$.

- The second case is g bounded from above and $m_r(h) \to 0$. First note that, since $|\Gamma^r| g^{1/2} \det h^{1/2} = 1$ and g is bounded from above, $\det h$ is bounded from below. It follows that $\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_n} = \operatorname{Vol}(\delta_r \cdot \mathbb{Z}^{2n} \backslash \mathbb{R}^{2n}, h) = \operatorname{Vol}(\mathbb{Z}^{2n} \backslash \mathbb{R}^{2n}, h[\delta_r]) \geqslant C'$ for some $C' > 0$. Since $\left\{1/4 \|X\|^2_{\mathbf{m}_\mathfrak{n}} \mid X \in \Gamma^r_\mathfrak{n}\right\} = \left\{1/4 h[\delta_r][X] \mid X \in \mathbb{Z}^{2n}\right\}$ we have the following estimate for the third term in the first line of (2.18):
$$\frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n} \int_1^\infty \sum_{X \in \Gamma_\mathfrak{n} \backslash \{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_\mathfrak{n}}}{4} t} t^{n-\tau-1} \mathrm{d}t \geqslant \frac{C'}{(4\pi)^n} \int_1^\infty \sum_{X \in \mathbb{Z}^{2n} \backslash \{0\}} e^{-\frac{h[\delta_r][X]}{4} t} t^{n-\tau-1} \mathrm{d}t\,.$$
Since $m_r(h) = m(h[\delta_r])$ the right hand side goes to $+\infty$ when $m_r(h) \to 0$ by Lemma 2.9.
- The last case is $g > a > 0$ and $d_n(h) \to \infty$. We have a look at the term
$$\frac{a_N^{\mathbf{m}}}{\tau - \dim G/2 + N}\,,$$
which is, as we have noted above, always positive and a linear combination of monomials of order $2N$ in $c_1^{\mathbf{m}}, \ldots, c_n^{\mathbf{m}}$. Since $c_j^{\mathbf{m}} = g^{1/2} d_j(h)$ and g is bounded from below, we have
$$\frac{a_N^{\mathbf{m}}}{\tau - \dim G/2 + N} \to \infty \quad \text{as } d_n(h) \to \infty\,.$$

$\square$

To understand the restrictions on the dimension of $\Gamma \backslash G$ and the position of s in Theorem 2.11*(i)* and *(ii)*, respectively, we introduce a deformation $\rho \mapsto \mathbf{m}_\rho$ of a Heisenberg-like metric $\mathbf{m}$ in the next proposition. With respect to the submersion structure of $(\Gamma \backslash G, \mathbf{m})$ from Proposition 1.10, this deformation is simply a scaling of the base and fibre tori $T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}$ and $T_{\mathfrak{z},\mathbf{m}_\mathfrak{z}}$. In a series of lemmata, we will then prove asymptotics for the various terms in the formulas for the ζ-function and the height as $\rho \to \infty$. In Theorem 2.19 we show that the restrictions on the dimension of $\Gamma \backslash G$ and the position of s in the context of Theorem 2.11 were indeed necessary.

PROPOSITION 2.12. *Let $(\mathfrak{g}, \mathbf{m})$ be a metric nonsingular Heisenberg-like Lie algebra, $\mathfrak{z}$ the centre of $\mathfrak{g}$ and $\mathfrak{n}$ the orthogonal complement of $\mathfrak{z}$ w.r.t $\mathbf{m}$. For real numbers $\sigma, \tau > 0$ define the metric $\mathbf{m}_{\sigma,\tau}$ by the requirement that $\mathfrak{z} \perp \mathfrak{n}$ and*
$$(\mathbf{m}_{\sigma,\tau})_\mathfrak{z} := \sigma \cdot \mathbf{m}_\mathfrak{z}\,,$$
$$(\mathbf{m}_{\sigma,\tau})_\mathfrak{n} := \tau \cdot \mathbf{m}_\mathfrak{n}\,.$$
Then $(\mathfrak{g}, \mathbf{m}_{\sigma,\tau})$ is Heisenberg-like, too. Moreover, if $0 < c_1^{\mathbf{m}} \leqslant \ldots \leqslant c_k^{\mathbf{m}}$ are the real numbers such that $\pm i c_j^{\mathbf{m}} \|Z\|_{\mathbf{m}}$ are the eigenvalues of $j_{\mathbf{m}}(Z)$, $Z \in \mathfrak{z}$, then $\pm i c_j^{\mathbf{m}_{\sigma,\tau}} \|Z\|_{\mathbf{m}_{\sigma,\tau}} = \pm i \sqrt{\sigma} \tau^{-1} c_j^{\mathbf{m}} \|Z\|_{\mathbf{m}_{\sigma,\tau}}$

are the eigenvales of $j_{\mathbf{m}_{\sigma,\tau}}(Z)$, $Z \in \mathfrak{z}$. *In particular, if* $\ell = \dim \mathfrak{z}$ *and* $2n = \dim \mathfrak{n}$ *then for any* $\rho > 0$ *the Lie algebra* $(\mathfrak{g}, \mathbf{m}_\rho)$ *with*

$$\mathbf{m}_\rho := \mathbf{m}_{\rho^{1/\ell}, \rho^{-1/2n}}$$

is Heisenberg-like too and $c_j^{\mathbf{m}_\rho} = \rho^{1/2\ell + 1/2n} \cdot c_j^{\mathbf{m}}$ *for all* $1 \leqslant j \leqslant n$.

PROOF. Let $Z \in \mathfrak{z}$ and $V, W \in \mathfrak{n}$ be arbitrary. By definition of $\mathbf{m}_{\sigma,\tau}$ and $j_{\mathbf{m}}$ (and $j_{\mathbf{m}_{\sigma,\tau}}$ resp.) (see (1.9)) we have

$$\begin{aligned}\tau \cdot \mathbf{m}\left(j_{\mathbf{m}_{\sigma,\tau}}(Z)V, W\right) &= \mathbf{m}_{\sigma,\tau}\left(j_{\mathbf{m}_{\sigma,\tau}}(Z)V, W\right) = \mathbf{m}_{\sigma,\tau}(Z, [V,W]) \\ &= \sigma \cdot \mathbf{m}(Z, [V,W]) = \sigma \cdot \mathbf{m}\left(j_{\mathbf{m}}(Z)V, W\right) .\end{aligned}$$

Since Z, V, W were arbitrary, this implies

$$j_{\mathbf{m}_{\sigma,\tau}} = \sigma \cdot \tau^{-1} j_{\mathbf{m}} .$$

For every $Z \in \mathfrak{z}$ we have $\mathbf{m}_{\sigma,\tau}(Z, Z) = \sigma \cdot \mathbf{m}(Z, Z)$ which means that the eigenvalues of $j_{\mathbf{m}_{\sigma,\tau}}(Z)$ are

$$\pm i\sigma\tau^{-1} c_j^{\mathbf{m}} \|Z\|_{\mathbf{m}} = \pm i\sqrt{\sigma}\tau^{-1} c_j^{\mathbf{m}} \|Z\|_{\mathbf{m}_{\sigma,\tau}} .$$

□

From here on, let G be a connected and simply connected nonsingular 2-step nilpotent Lie group and $\Gamma \subset G$ a uniform subgroup. Assume that there exists $[\mathbf{m}] \in \mathscr{S}\mathscr{M}^{HL}(\Gamma, G)$.

REMARK 2.13. The $\mathbf{m}_\rho$-dual lattice to $\Gamma_{\mathfrak{n}}$ is given by

$$\begin{aligned}\Gamma_{\mathfrak{n}}^*(\rho) &= \left\{\lambda \in \mathfrak{n} \,|\, \langle \lambda, X \rangle_{(\mathbf{m}_\rho)_{\mathfrak{n}}} \in \mathbb{Z}\, \forall X \in \Gamma_{\mathfrak{n}}\right\} = \left\{\lambda \in \mathfrak{n} \,|\, \rho^{-1/2n}\langle \lambda, X \rangle_{\mathbf{m}_{\mathfrak{n}}} \in \mathbb{Z}\, \forall X \in \Gamma_{\mathfrak{n}}\right\} \\ &= \left\{\rho^{1/2n}\lambda \in \mathfrak{n} \,|\, \langle \lambda, X \rangle_{\mathbf{m}_{\mathfrak{n}}} \in \mathbb{Z}\, \forall X \in \Gamma_{\mathfrak{n}}\right\} = \rho^{1/2n} \cdot \Gamma_{\mathfrak{n}}^* ,\end{aligned}$$

where $\Gamma_{\mathfrak{n}}^*$ is the usual $\mathbf{m}$-dual to $\Gamma_{\mathfrak{n}}$. From this we obtain

$$\left\{\|\lambda\|_{\mathbf{m}_\rho}^2 \,|\, \lambda \in \Gamma_{\mathfrak{n}}^*(\rho)\right\} = \left\{\rho^{1/n}\|\lambda\|_{\mathbf{m}_\rho}^2 \,|\, \lambda \in \Gamma_{\mathfrak{n}}^*\right\} = \rho^{1/2n} \cdot \left\{\|\lambda\|_{\mathbf{m}}^2 \,|\, \lambda \in \Gamma_{\mathfrak{n}}^*\right\} . \tag{2.19}$$

Analoguosly, we have $\Gamma_{\mathfrak{z}}^*(\rho) = \rho^{-1/\ell}\Gamma_{\mathfrak{z}}^*$ for the $\mathbf{m}_\rho$-dual $\Gamma_{\mathfrak{z}}^*(\rho)$ to $\Gamma_{\mathfrak{z}}$ and

$$\left\{\|\lambda\|_{\mathbf{m}_\rho}^2 \,|\, \lambda \in \Gamma_{\mathfrak{z}}^*(\rho)\right\} = \left\{\rho^{-2/\ell}\|\lambda\|_{\mathbf{m}_\rho}^2 \,|\, \lambda \in \Gamma_{\mathfrak{z}}^*\right\} = \rho^{-1/\ell} \cdot \left\{\|\lambda\|_{\mathbf{m}}^2 \,|\, \lambda \in \Gamma_{\mathfrak{z}}^*\right\} . \tag{2.20}$$

For the next lemma, recall the definition of $s_{V,t}^{\mathbf{m}}$ in (1.42):

$$s_{V,t}^{\mathbf{m}} : \mathfrak{z} \ni \lambda \mapsto e^{-4\pi^2\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}}^2 t} \prod_{j=1}^{n} \frac{2\pi\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|\lambda\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t\right)} \in \mathbb{R} .$$

LEMMA 2.14. *Let $s \in [0,\infty)$ and assume there exists $[\mathbf{m}] \in \mathscr{S}\mathscr{M}^{HL}(\Gamma, G)$. Then*

$$\frac{\operatorname{Vol} T_{\mathfrak{n},(\mathbf{m}_\rho)_\mathfrak{n}}}{(4\pi)^n} \int_1^\infty \sum_{\lambda \in \Gamma_\mathfrak{z}^*(\rho)\setminus\{0\}} s_{V,t}^{\mathbf{m}_\rho}(\lambda) t^{s-n-1} \mathrm{d}t \to 0 \quad as \quad \rho \to \infty\,.$$

PROOF. Let $\rho \geqslant 1$. With formula (2.20) we calculate straight forwardly:

$$\begin{aligned}
&\frac{\operatorname{Vol} T_{\mathfrak{n},(\mathbf{m}_\rho)_\mathfrak{n}}}{(4\pi)^n} \int_1^\infty \sum_{\lambda \in \Gamma_\mathfrak{z}^*(\rho)\setminus\{0\}} s_{V,t}^{\mathbf{m}_\rho}(\lambda) t^{s-n-1} \mathrm{d}t \\
&\quad = \frac{\operatorname{Vol} T_{\mathfrak{n},(\mathbf{m}_\rho)_\mathfrak{n}}}{(4\pi)^n} \int_1^\infty \sum_{\lambda \in \Gamma_\mathfrak{z}^*(\rho)\setminus\{0\}} e^{-4\pi^2 \|\lambda\|^2_{\mathbf{m}_\rho} t} \prod_{j=1}^n \frac{2\pi\|\lambda\|_{\mathbf{m}_\rho} c_j^{\mathbf{m}_\rho} t}{\sinh\left(2\pi\|\lambda\|_{\mathbf{m}_\rho} c_j^{\mathbf{m}_\rho} t\right)} t^{s-n-1}\mathrm{d}t \\
&= \rho^{-1/2} \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n} \int_1^\infty \sum_{\lambda \in \Gamma_\mathfrak{z}^*\setminus\{0\}} e^{-4\pi^2 \rho^{-1/\ell} \|\lambda\|^2_{\mathbf{m}} t} \prod_{j=1}^n \frac{2\pi\rho^{-1/2\ell}\|\lambda\|_{\mathbf{m}} \rho^{1/2\ell+1/2n} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\rho^{-1/2\ell}\|\lambda\|_{\mathbf{m}} \rho^{1/2\ell+1/2n} c_j^{\mathbf{m}} t\right)} t^{s-n-1}\mathrm{d}t \\
&\quad = \rho^{-s/2n} \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n} \int_{\rho^{1/2n}}^\infty \sum_{\lambda \in \Gamma_\mathfrak{z}^*\setminus\{0\}} e^{-4\pi^2 \rho^{-1/\ell-1/2n} \|\lambda\|^2_{\mathbf{m}} t} \prod_{j=1}^n \frac{2\pi\|\lambda\|_{\mathbf{m}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|\lambda\|_{\mathbf{m}} c_j^{\mathbf{m}} t\right)} t^{s-n-1}\mathrm{d}t \\
&\qquad \leqslant \rho^{-s/2n} \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n} \int_{\rho^{1/2n}}^\infty \sum_{\lambda \in \Gamma_\mathfrak{z}^*\setminus\{0\}} \prod_{j=1}^n \frac{2\pi\|\lambda\|_{\mathbf{m}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|\lambda\|_{\mathbf{m}} c_j^{\mathbf{m}} t\right)} t^{s-n-1}\mathrm{d}t \to 0
\end{aligned}$$

as $\rho \to \infty$. ☐

Now recall the definition $\sigma_X^{\mathbf{m}}$ in Theorem 1.78*(ii)*:

$$\begin{aligned}
\sigma_X^{\mathbf{m}}(t) &= (4\pi t)^{\ell/2} \widehat{s_{V,t}^{\mathbf{m}}}(X) \\
&= (4\pi t)^{\ell/2} \int_\mathfrak{z} e^{-2\pi i \langle X,\xi\rangle_{\mathbf{m}_\mathfrak{z}}} e^{-4\pi^2\|X\|^2_{\mathbf{m}_\mathfrak{z}} t} \prod_{j=1}^n \frac{2\pi\|X\|_{\mathbf{m}_\mathfrak{z}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|X\|_{\mathbf{m}_\mathfrak{z}} c_j^{\mathbf{m}} t\right)} \mathrm{dVol}_\mathfrak{z}(\xi)\,.
\end{aligned}$$

Also, recall from Corollary 1.84 that $\sigma_X^{\mathbf{m}}$ is nonnegative.

LEMMA 2.15. *Let $s \in [0,\infty)$ and assume there exists $[\mathbf{m}] \in \mathscr{S}\mathscr{M}^{HL}(\Gamma, G)$. Let $\psi : [1,\infty) \to \mathbb{R}$ be defined by*

$$\psi(\rho) := \int_0^1 \sum_{X \in \Gamma_\mathfrak{z}\setminus\{0\}} \sigma_X^{\mathbf{m}_\rho}(t) t^{s-\dim G/2-1} \mathrm{d}t\,.$$

Then ψ is bounded.

PROOF. By Proposition 2.12 we have $c_j^{\mathbf{m}_\rho} = c_j^{\mathbf{m}} \cdot \rho^{1/2\ell+1/2n}$ and $\|X\|_{(\mathbf{m}_\rho)_\mathfrak{z}} = \|X\|_{\mathbf{m}_\mathfrak{z}} \cdot \rho^{1/2\ell}$ for all $X \in \mathfrak{z}$. Hence

$$\begin{aligned}
&\sigma_X^{\mathbf{m}_\rho}(t)\\
&= (4\pi t)^{\ell/2} \int_{\mathfrak{z}} e^{-2\pi i \langle X,\xi\rangle_{\mathbf{m}_{\mathfrak{z}}}\rho^{1/\ell}} e^{-4\pi^2\|\xi\|^2_{\mathbf{m}_{\mathfrak{z}}}\rho^{1/\ell}t} \prod_{j=1}^n \frac{2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} \rho^{1/\ell+1/2n} t}{\sinh\left(2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} \rho^{1/\ell+1/2n} t\right)} \rho^{1/2}\, \mathrm{dVol}_{(\mathfrak{z},\mathbf{m}_{\mathfrak{z}})}(\xi)\\
&= (4\pi t)^{\ell/2}\rho^{-1/2} \int_{\mathfrak{z}} e^{-2\pi i \langle X,\xi\rangle_{\mathbf{m}_{\mathfrak{z}}}} e^{-4\pi^2\|\xi\|^2_{\mathbf{m}_{\mathfrak{z}}}\rho^{-1/\ell}t} \prod_{j=1}^n \frac{2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} \rho^{1/2n} t}{\sinh\left(2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} \rho^{1/2n} t\right)}\, \mathrm{dVol}_{(\mathfrak{z},\mathbf{m}_{\mathfrak{z}})}(\xi)\,,
\end{aligned}$$

where we have transformed the integral via the diffeomorphism $\mathfrak{z} \ni \xi \mapsto \rho^{-1/\ell}\xi \in \mathfrak{z}$. From this we obtain

$$\begin{aligned}
&\frac{1}{(4\pi)^{\ell/2}}\psi(\rho)\\
&= \rho^{-1/2}\int_0^1 \sum_{X\in\Gamma_{\mathfrak{z}}\setminus\{0\}} \int_{\mathfrak{z}} e^{-2\pi i \langle X,\xi\rangle_{\mathbf{m}_{\mathfrak{z}}}} e^{-4\pi^2\|\xi\|^2_{\mathbf{m}_{\mathfrak{z}}}\rho^{-1/\ell}t} \prod_{j=1}^n \frac{2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} \rho^{1/2n} t}{\sinh\left(2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} \rho^{1/2n} t\right)}\, \mathrm{dVol}_{\mathfrak{z}}(\xi)\, t^{s-n-1}\mathrm{d}t\\
&= \rho^{-s/2n}\int_0^{\rho^{1/2n}} \sum_{X\in\Gamma_{\mathfrak{z}}\setminus\{0\}} \int_{\mathfrak{z}} e^{-2\pi i \langle X,\xi\rangle_{\mathbf{m}_{\mathfrak{z}}}} e^{-\frac{4\pi^2\|\xi\|^2_{\mathbf{m}_{\mathfrak{z}}} t}{\rho^{1/\ell+1/2n}}} \prod_{j=1}^n \frac{2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t\right)}\, \mathrm{dVol}_{\mathfrak{z}}(\xi)\, t^{s-n-1}\mathrm{d}t
\end{aligned} \tag{2.21}$$

We split the outer integral into $\int_0^{\rho^{1/2n}} = \int_0^1 + \int_1^{\rho^{1/2n}}$ and treat $\int_0^1$ first. Our claim is that $\lim_{\rho\to\infty}\int_0^1$ converges to a positive real number. To show this we have to exchange $\lim_{\rho\to\infty}$ with 3 barriers, an integral, a sum and another integral. The innermost integral does not pose a problem. Indeed, we have for all $t \in (0,1]$:

$$\left| e^{-2\pi i \langle X,\xi\rangle_{\mathbf{m}_{\mathfrak{z}}}} e^{-\frac{4\pi^2\|\xi\|^2_{\mathbf{m}_{\mathfrak{z}}} t}{\rho^{1/\ell+1/2n}}} \prod_{j=1}^n \frac{2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t\right)} \right| \leqslant \prod_{j=1}^n \frac{2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t\right)}$$

for all $\rho \in [1,\infty)$. The right hand side is integrable. Theorem 1.36 yields

$$\begin{aligned}
&\lim_{\rho\to\infty}\int_{\mathfrak{z}} e^{-2\pi i \langle X,\xi\rangle_{\mathbf{m}_{\mathfrak{z}}}} e^{-\frac{4\pi^2\|\xi\|^2_{\mathbf{m}_{\mathfrak{z}}} t}{\rho^{1/\ell+1/2n}}} \prod_{j=1}^n \frac{2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t\right)}\, \mathrm{dVol}_{\mathfrak{z}}(\xi)\\
&\qquad\qquad = \int_{\mathfrak{z}} e^{-2\pi i \langle X,\xi\rangle_{\mathbf{m}_{\mathfrak{z}}}} \prod_{j=1}^n \frac{2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t\right)}\, \mathrm{dVol}_{\mathfrak{z}}(\xi)\,.
\end{aligned}$$

Note that the right hand side is positive by Lemma 1.76 and Proposition 1.29*(xi)*.

To be able to exchange $\lim_{\rho\to\infty}$ with the remaining series and integral, we first prove an estimate for the inner integral. Define

$$I(X,t,\rho) := \int_{\mathfrak{z}} e^{-2\pi i \langle X,\xi\rangle_{\mathbf{m}_{\mathfrak{z}}}} e^{-\frac{4\pi^2\|\xi\|^2_{\mathbf{m}_{\mathfrak{z}}} t}{\rho^{1/\ell+1/2n}}} \prod_{j=1}^n \frac{2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t\right)}\, \mathrm{dVol}_{\mathfrak{z}}(\xi)\,.$$

LEMMA 2.16. *For each $N \in \mathbb{N}$ there exists $C > 0$ such that*

$$|I(X,t,\rho)| \leqslant C \cdot \frac{t^N}{\|X\|_{\mathbf{m}}^{2(N+\ell)}}$$

for all $X \in \Gamma_{\mathfrak{z}}\backslash\{0\}$, $t \in (0,1]$ and $\rho \in [1,\infty)$.

PROOF. Choose an $\mathbf{m}_{\mathfrak{z}}$-orthonormal basis of $\mathfrak{z}$ and write $\xi = (\xi_1, \dots, \xi_\ell)$ w.r.t. this basis. Let $\rho \geqslant 1$, $t \in (0,1]$ and $X \in \Gamma_{\mathfrak{z}}\backslash\{0\}$. By repeated integration by parts, we obtain for every $1 \leqslant j \leqslant \ell$ and every $k \in \mathbb{N}$, writing $\|\xi\|$ for $\|\xi\|_{\mathbf{m}_{\mathfrak{z}}}$:

$$(-2\pi i X_j)^k I(X,t,\rho) = (-1)^k \int_{\mathfrak{z}} e^{-2\pi i \langle X,\xi\rangle} \frac{\partial^k}{\partial \xi_j^k} \left(e^{-\frac{4\pi^2\|\xi\|^2 t}{\rho^{1/\ell+1/2n}}} \prod_{j=1}^{n} \frac{2\pi\|\xi\| c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|\xi\| c_j^{\mathbf{m}} t\right)} \right) \mathrm{d}\xi\,.$$

Hence,

$$|2\pi X_j|^k \cdot |I(X,t,\rho)| \leqslant \int_{\mathfrak{z}} \left| \frac{\partial^k}{\partial \xi_j^k} \left(e^{-\frac{4\pi^2\|\xi\|^2 t}{\rho^{1/\ell+1/2n}}} \prod_{j=1}^{n} \frac{2\pi\|\xi\| c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|\xi\| c_j^{\mathbf{m}} t\right)} \right) \right| \mathrm{d}\xi\,. \tag{2.22}$$

Note that the derivative in (2.22) is a linear combination of functions of the form

$$\mathfrak{z} \ni \xi \mapsto t^{m/2}\rho^{-\beta} f\left(\sqrt{t}\tfrac{\xi}{\rho^{1/2\ell+1/4n}}\right) \cdot g(t\xi) \in \mathbb{R} \text{ with } m \in \{k, k+1, \dots, 2k\}\,,$$

where $\beta \geqslant 0$ and $f, g \in \mathscr{S}(\mathfrak{z})$. Since $\mathscr{S}(\mathfrak{z}) \subset L^1(\mathfrak{z})$ we have for all $k \in \mathbb{N}$ with $k \geqslant 2\ell$:

$$\begin{aligned} \int_{\mathfrak{z}} \left| t^{k/2}\rho^{-\beta} f\left(\sqrt{t}\tfrac{\xi}{\rho^{1/\ell+1/2n}}\right) \cdot g(t\xi)\right| \mathrm{d}\xi &= t^{k/2-\ell}\rho^{-\beta} \int_{\mathfrak{z}} \left| f\left(\tfrac{\xi}{\rho^{1/\ell+1/2n} t}\right) \cdot g(\xi)\right| \mathrm{d}\xi \\ \leqslant t^{k/2-\ell} \int_{\mathfrak{z}} |\max f| \cdot |g(\xi)|\, \mathrm{d}\xi &\leqslant \int_{\mathfrak{z}} |\max f| \cdot |g(\xi)|\, \mathrm{d}\xi\,. \end{aligned}$$

This and (2.22) imply that there exists $C_{j,k} > 0$ such that

$$|X_j|^k \cdot |I(X,t,\rho)| \leqslant C_{j,k} \cdot t^{k/2-\ell}\,.$$

We sum over j and obtain

$$|I(X,t,\rho)| \leqslant \sum_{j=1}^{\ell} C_{j,k} \cdot \frac{t^{k/2-\ell}}{\|X\|_k^k}\,.$$

Now let $N \in \mathbb{N}$. Choose $k := 2(N+\ell)$ and let $C > 0$ be such that $\|y\|_k^{-k} \sum_{j=1}^{\ell} C_{j,k} \leqslant C\|y\|^{-k}$ for all $y \in \mathfrak{z}\backslash\{0\}$. Then we have

$$|I(X,t,\rho)| \leqslant C \cdot \frac{t^N}{\|X\|^{2(N+\ell)}}\,.$$

$\square$

We return to the proof of

$$\lim_{\rho\to\infty}\int_0^1 \sum_{X\in\Gamma_{\mathfrak{z}}\setminus\{0\}} I(X,t,\rho)\cdot t^{s-n-1}\mathrm{d}t = \int_0^1 \sum_{X\in\Gamma_{\mathfrak{z}}\setminus\{0\}} \lim_{\rho\to\infty} I(X,t,\rho)\cdot t^{s-n-1}\mathrm{d}t < \infty\,. \tag{2.23}$$

Choose a basis of $\Gamma_{\mathfrak{z}}$ and let Y be the Gram matrix of $\Gamma_{\mathfrak{z}}$ w.r.t. this basis. Choose $N\in\mathbb{N}$ with $N+s-n-1>-1$. By the last lemma there exists $C>0$ such that

$$\begin{aligned}\sum_{X\in\Gamma_{\mathfrak{z}}\setminus\{0\}}\sup_{\rho\in[1,\infty)}|I(X,t,\rho)| \leqslant C\cdot t^N \sum_{X\in\Gamma_{\mathfrak{z}}\setminus\{0\}}\frac{1}{\|X\|_2^{2(N+\ell)}} &= C\cdot t^N\sum_{a\in\mathbb{Z}^\ell\setminus\{0\}}\frac{1}{(Y[a])^{2(N+\ell)}}\\ &= C\cdot t^N\cdot\zeta_{Ep}(Y,2(N+\ell))\,,\end{aligned}$$

where ζ_{Ep} is Epstein's ζ-function, see Definition 2.33. It follows that

$$\sum_{X\in\Gamma_{\mathfrak{z}}\setminus\{0\}} I(X,t,\rho)$$

converges normally on $(t,\rho)\in[0,1]\times[1,\infty)$ and that

$$\sum_{X\in\Gamma_{\mathfrak{z}}\setminus\{0\}}|I(X,t,\rho)|\cdot t^{s-n-1}\leqslant C\cdot t^{N+s-n-1}\,.$$

The right hand side is integrable on $t\in[0,1]$. Now (2.23) follows from Theorem 1.36 and the continuity of normally convergent series.

Let us bound the integral $\int_1^{\rho^{1/2n}}$ in (2.21). We use Poisson's summation formula (see Corollary 1.35). Keeping in mind that the Fourier transform is an involution on even functions, we calculate

$$\begin{aligned}
&\int_1^{\rho^{1/2n}}\sum_{X\in\Gamma_{\mathfrak{z}}\setminus\{0\}}\int_{\mathfrak{z}} e^{-2\pi i\langle X,\xi\rangle_{\mathbf{m}_{\mathfrak{z}}}} e^{-\frac{4\pi^2\|\xi\|^2_{\mathbf{m}_{\mathfrak{z}}}t}{\rho^{1/\ell+1/2n}}}\prod_{j=1}^n\frac{2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}}c_j^{\mathbf{m}}t}{\sinh\left(2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}}c_j^{\mathbf{m}}t\right)}\,\mathrm{dVol}_{\mathfrak{z}}(\xi)t^{s-n-1}\mathrm{d}t\\
&=\int_1^{\rho^{1/2n}}\sum_{X\in\Gamma_{\mathfrak{z}}}\int_{\mathfrak{z}} e^{-2\pi i\langle X,\xi\rangle_{\mathbf{m}_{\mathfrak{z}}}} e^{-\frac{4\pi^2\|\xi\|^2_{\mathbf{m}_{\mathfrak{z}}}t}{\rho^{1/\ell+1/2n}}}\prod_{j=1}^n\frac{2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}}c_j^{\mathbf{m}}t}{\sinh\left(2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}}c_j^{\mathbf{m}}t\right)}\,\mathrm{dVol}_{\mathfrak{z}}(\xi)t^{s-n-1}\mathrm{d}t\\
&\qquad-\int_1^{\rho^{1/2n}}\int_{\mathfrak{z}} e^{-\frac{4\pi^2\|\xi\|^2_{\mathbf{m}_{\mathfrak{z}}}t}{\rho^{1/\ell+1/2n}}}\prod_{j=1}^n\frac{2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}}c_j^{\mathbf{m}}t}{\sinh\left(2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}}c_j^{\mathbf{m}}t\right)}\,\mathrm{dVol}_{\mathfrak{z}}(\xi)t^{s-n-1}\mathrm{d}t\\
&=\mathrm{Vol}(\Gamma_{\mathfrak{z}}^*\backslash\mathfrak{z})\int_1^{\rho^{1/2n}}\sum_{X\in\Gamma_{\mathfrak{z}}^*} e^{-\frac{4\pi^2\|X\|^2_{\mathbf{m}_{\mathfrak{z}}}t}{\rho^{1/\ell+1/2n}}}\prod_{j=1}^n\frac{2\pi\|X\|_{\mathbf{m}_{\mathfrak{z}}}c_j^{\mathbf{m}}t}{\sinh\left(2\pi\|X\|_{\mathbf{m}_{\mathfrak{z}}}c_j^{\mathbf{m}}t\right)}\,\mathrm{dVol}_{\mathfrak{z}}(\xi)t^{s-n-1}\mathrm{d}t\\
&\qquad-\int_1^{\rho^{1/2n}}\int_{\mathfrak{z}} e^{-\frac{4\pi^2\|\xi\|^2_{\mathbf{m}_{\mathfrak{z}}}t}{\rho^{1/\ell+1/2n}}}\prod_{j=1}^n\frac{2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}}c_j^{\mathbf{m}}t}{\sinh\left(2\pi\|\xi\|_{\mathbf{m}_{\mathfrak{z}}}c_j^{\mathbf{m}}t\right)}\,\mathrm{dVol}_{\mathfrak{z}}(\xi)t^{s-n-1}\mathrm{d}t
\end{aligned}$$

$$\leqslant \mathrm{Vol}(\Gamma_{\mathfrak{z}}^*\backslash\mathfrak{z}) \int\limits_1^{\rho^{1/2n}} \sum_{X\in\Gamma_{\mathfrak{z}}^*\backslash\{0\}} e^{-\frac{4\pi^2\|X\|^2_{\mathbf{m}_{\mathfrak{z}}} t}{\rho^{1/\ell+1/2n}}} \prod_{j=1}^{n} \frac{2\pi\|X\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|X\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t\right)} \mathrm{dVol}_{\mathfrak{z}}(\xi) t^{s-n-1}\mathrm{d}t$$

$$+ \mathrm{Vol}(\Gamma_{\mathfrak{z}}^*\backslash\mathfrak{z}) \int\limits_1^{\rho^{1/2n}} t^{s-n-1}\mathrm{d}t$$

$$\leqslant \mathrm{Vol}(\Gamma_{\mathfrak{z}}^*\backslash\mathfrak{z}) \int\limits_1^{\infty} \sum_{X\in\Gamma_{\mathfrak{z}}^*\backslash\{0\}} \prod_{j=1}^{n} \frac{2\pi\|X\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t}{\sinh\left(2\pi\|X\|_{\mathbf{m}_{\mathfrak{z}}} c_j^{\mathbf{m}} t\right)} \mathrm{dVol}_{\mathfrak{z}}(\xi) t^{s-n-1}\mathrm{d}t$$

$$+ \mathrm{Vol}(\Gamma_{\mathfrak{z}}^*\backslash\mathfrak{z}) \begin{cases} \frac{1}{s-n}\left(\rho^{s/2n-1/2}-1\right), & \text{for } s\neq n\,,\\ \frac{1}{2n}\ln\rho, & \text{for } s=n\,. \end{cases}$$

The term $\mathrm{Vol}(\Gamma_{\mathfrak{z}}^*\backslash\mathfrak{z}) \int\limits_1^{\infty}$ is finite by Remark 1.92 and Lemma 2.3. From this calculation and formulas (2.21) and (2.23) it follows that

$$\frac{1}{(4\pi)^{\ell/2}}\psi(\rho) \leqslant \rho^{-s/2n} \cdot \begin{cases} C' + \frac{v}{s-n}\left(\rho^{s/2n-1/2}-1\right), & \text{for } s\neq n\,,\\ C' + v\cdot\ln\rho, & \text{for } s=n \end{cases}, \tag{2.24}$$

where $v = \mathrm{Vol}(\Gamma_{\mathfrak{z}}^*\backslash\mathfrak{z})$ and $C' > 0$ is a constant. The right hand side of (2.24) is clearly bounded. □

We remind ourselves that we split the spectral ζ-function into two terms:

$$\zeta((\Gamma\backslash G, \mathbf{m}_\rho), s) = \zeta_B((\Gamma\backslash G, \mathbf{m}_\rho), s) + \zeta_F((\Gamma\backslash G, \mathbf{m}_\rho), s)\,,$$

see Theorem 2.5.

LEMMA 2.17. *Let $s \in [0,\infty)$ and asume there exists $[\mathbf{m}] \in \mathscr{S}\mathscr{M}^{HL}(\Gamma, G)$. Then,*

(i) $\zeta_B((\Gamma\backslash G, \mathbf{m}_\rho), s) \in O(1)$ and $\zeta_B'((\Gamma\backslash G, \mathbf{m}_\rho), 0) \in O(\ln\rho)$ as $\rho\to\infty$,

(ii) $\zeta_B((\Gamma\backslash G, \mathbf{m}_\rho), s) \to +\infty$ if $s>0$, and $\zeta_B'((\Gamma\backslash G, \mathbf{m}_\rho), 0) \to +\infty$ as $\rho\to 0$.

PROOF. Choose an $\mathbf{m}_{\mathfrak{n}}$-orthonormal basis of $\mathfrak{n}$ and identify $\mathfrak{n}$ by a linear isometry with $(\mathbb{R}^{2n}, \langle\cdot,\cdot\rangle_{std})$. Furthermore, choose a matrix $M \in M(2n;\mathbb{R})$ such that $\Gamma_{\mathfrak{n}} = M\cdot\mathbb{Z}^{2n}$ and let $Y = \mathrm{Id}[M] = {}^{\mathrm{t}}M\cdot M$ be the corresponding Gram matrix. Note that $\rho^{-1/2n}Y$ is the Gram matrix of $\Gamma_{\mathfrak{n}}$ w.r.t. the inner product $(\mathbf{m}_\rho)_{\mathfrak{n}}$. By Observation 2.38 we have

$$\Gamma(s)\zeta_B((\Gamma\backslash G, \mathbf{m}_\rho), s) = \pi^{-s}\Gamma(s)\zeta_{Ep}(4\pi\rho^{1/2n}Y^{-1}, s) - \frac{\mathrm{Vol}\, T_{\mathfrak{n},(\mathbf{m}_\rho)_{\mathfrak{n}}}}{(4\pi)^n}\frac{1}{s-n}\,,$$

where ζ_{Ep} is Epstein's ζ-function, see Definition 2.33. Note that the pole of the last term is canceled by the pole of ζ_{Ep}, see Theorem 2.36. By the very definition of ζ_{Ep} we have

$$\pi^{-s}\zeta_{Ep}(4\pi\rho^{1/2n}Y^{-1},s) - \frac{\operatorname{Vol} T_{\mathfrak{n},(\mathbf{m}_\rho)_\mathfrak{n}}}{(4\pi)^n}\frac{1}{\Gamma(s)\cdot(s-n)} = \rho^{-s/2n}\zeta_{Ep}(4\pi^2 Y^{-1},s) - \rho^{-1/2}\frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n}\frac{1}{\Gamma(s)\cdot(s-n)},$$

which is bounded as $\rho \to \infty$ and which goes to $+\infty$ as $\rho \to 0$. Furthermore, differentiating this in $s = 0$ yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\left(\rho^{-s/2n}\zeta_{Ep}(4\pi^2Y^{-1},s) - \rho^{-1/2}\frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n}\frac{1}{\Gamma(s)\cdot(s-n)}\right)$$
$$= -\frac{1}{2n}\ln\rho\cdot\zeta_{Ep}(4\pi^2Y^{-1},0) + \zeta'_{Ep}(4\pi^2Y^{-1},0) + \frac{\rho^{-1/2}}{n}\frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n},$$

since $1/\Gamma(s) = \sum_{j=1}^{\infty} b_j s^j$ with $b_1 = 1$, see **[OLBC10**, 5.7.1]. This is in $O(\ln\rho)$ as $\rho \to \infty$ and in $O(\rho^{-1/2})$ as $\rho \to 0$. □

LEMMA 2.18. *Let $s \in [0, \dim G/2)\setminus \operatorname{sing}(\dim G)$ and assume that there exists $[\mathbf{m}] \in \mathscr{SM}^{HL}(\Gamma, G)$. Let $N \in \mathbb{N}_0$ be the smallest number such that $N > \dim G/2 - s - 1$. Then there exists $C > 0$ such that*

$$\int_0^1 \left(\sigma_0^{\mathbf{m}_\rho}(t) - T^N_{+0}[\sigma_0^{\mathbf{m}_\rho}](t)\right) t^{s-\dim G/2-1}\mathrm{d}t \quad\sim\quad (-1)^{N+1}\cdot C\cdot \rho^{(1/\ell+1/n)(\ell/2+n-s)}\cdot A(\rho)$$

as $\rho \to \infty$, where

$$A(\rho) := \begin{cases} 1, & \text{if } \dim G \text{ is odd or } s > 0\,,\\ \ln\rho, & \text{if } \dim G \text{ is even and } s = 0\,.\end{cases}$$

PROOF. Let $\rho > 1$. By Proposition 2.12 we have $c_j^{\mathbf{m}_\rho} = c_j^{\mathbf{m}}\cdot\rho^{1/2\ell+1/2n}$. Hence, with the expression for $\sigma_0^{\mathbf{m}}$ given in Remark 1.80, we obtain

$$\begin{aligned}\sigma_0^{\mathbf{m}_\rho}(t) &= \pi^{-\ell/2}\operatorname{Vol} S^{\ell-1}\int_0^\infty e^{-r^2}\prod_{j=1}^n \frac{\sqrt{t}c_j^{\mathbf{m}_\rho}r}{\sinh\left(\sqrt{t}c_j^{\mathbf{m}_\rho}r\right)}r^{\ell-1}\mathrm{d}r\\ &= \pi^{-\ell/2}\operatorname{Vol} S^{\ell-1}\int_0^\infty e^{-r^2}\prod_{j=1}^n \frac{\sqrt{t}c_j^{\mathbf{m}}\rho^{1/2\ell+1/2n}r}{\sinh\left(\sqrt{t}c_j^{\mathbf{m}}\rho^{1/2\ell+1/2n}r\right)}r^{\ell-1}\mathrm{d}r\\ &= \pi^{-\ell/2}\operatorname{Vol} S^{\ell-1}\int_0^\infty e^{-r^2}\prod_{j=1}^n \frac{\sqrt{\rho^{1/\ell+1/n}t}c_j^{\mathbf{m}}r}{\sinh\left(\sqrt{\rho^{1/\ell+1/n}t}c_j^{\mathbf{m}}r\right)}r^{\ell-1}\mathrm{d}r = \sigma_0^{\mathbf{m}}\left(\rho^{1/\ell+1/n}t\right).\end{aligned}$$

It follows that $T^N_{+0}[\sigma_0^{\mathbf{m}_\rho}](t) = T^N_{+0}[\sigma_0^{\mathbf{m}}](\rho^{1/\ell+1/n}t)$ and

$$\int_0^1 \left(\sigma_0^{\mathbf{m}_\rho}(t) - T^N_{+0}[\sigma_0^{\mathbf{m}_\rho}](t)\right) t^{s-\dim G/2-1}\mathrm{d}t$$

$$= \int_0^1 \left(\sigma_0^{\mathbf{m}}(\rho^{1/\ell+1/n} t) - T_{+0}^N[\sigma_0^{\mathbf{m}}](\rho^{1/\ell+1/n} t) \right) t^{s-\dim G/2-1} \mathrm{d}t$$

$$= \rho^{(1/\ell+1/n)(\dim G/2 - s)} \int_0^{\rho^{1/\ell+1/n}} \left(\sigma_0^{\mathbf{m}}(t) - T_{+0}^N[\sigma_0^{\mathbf{m}}](t) \right) t^{s-\dim G/2-1} \mathrm{d}t \,. \tag{2.25}$$

By Taylor's Theorem with Lagrange remainder we have

$$\left| \sigma_0^{\mathbf{m}}(t) - T_{+0}^N[\sigma_0^{\mathbf{m}}](t) \right| t^{s-\dim G/2-1} \leqslant \tfrac{1}{(N+1)!} \cdot (\sigma_0^{\mathbf{m}})^{(N+1)}(\tau_t) \cdot t^{N+s-\dim G/2}$$

for some $\tau_t \in (0,t)$. Since $N + s - \dim G/2 > -1$, the integral in (2.25) is finite. We investigate its behaviour as $\rho \to \infty$.

By Lemma 1.88, $\sigma_0^{\mathbf{m}}$ and all its derivatives are bounded on $[0, \infty)$. Hence, the integrand of (2.25) is in $O\left(t^{N+s-\dim G/2-1}\right)$ as $t \to \infty$. By choice of N we have $-1 \geqslant N + s - \dim G/2 - 1 > -2$. Equality on the left hand side occurs precisely if $N + s = \dim G/2$. Suppose $\dim G$ is odd. Since $N \in \mathbb{N}_0$, $2s$ is odd, i.e., $s \in \operatorname{sing}(\dim G)$, which is a contradiction. Now suppose $\dim G$ is even. Then $\dim G/2 \in \mathbb{N}$ and hence $s \in \mathbb{N}_0$. Since $s \in [0, \dim G/2) \setminus \operatorname{sing}(\dim G)$, we have $s = 0$. In summary, we have $N + s - \dim G/2 - 1 < -1$ unless $s = 0$ and $\dim G$ is even, in which case we have $N + s - \dim G/2 - 1 = -1$. It follows that in case $\dim G$ is odd or $s > 0$ we have

$$\lim_{\rho \to \infty} \int_0^{\rho^{1/\ell+1/n}} \left(\sigma_0^{\mathbf{m}}(t) - T_{+0}^N[\sigma_0^{\mathbf{m}}](t) \right) t^{s-\dim G/2-1} \mathrm{d}t = \int_0^{\infty} \left(\sigma_0^{\mathbf{m}}(t) - T_{+0}^N[\sigma_0^{\mathbf{m}}](t) \right) t^{s-\dim G/2-1} \mathrm{d}t \,,$$

which is finite, nonzero and has sign $(-1)^{N+1}$ by Corollary 1.89. On the other hand, if $\dim G$ is even and $s = 0$ we have

$$\int_0^{\rho^{1/\ell+1/n}} \left(\sigma_0^{\mathbf{m}}(t) - T_{+0}^N[\sigma_0^{\mathbf{m}}](t) \right) t^{s-\dim G/2-1} \mathrm{d}t \sim (-1)^{N+1} \cdot C_1 \cdot \ln \rho \text{ as } \rho \to \infty$$

for some $C_1 > 0$. These two facts together with (2.25) and $\dim G = \ell + 2n$ imply the desired result. □

THEOREM 2.19. *Let G be a connected and simply connected nonsingular 2-step nilpotent Lie group and $\Gamma \subset G$ a uniform subgroup. Assume there exists $[\mathbf{m}] \in \mathscr{S}\mathscr{M}^{HL}(\Gamma, G)$ and let $s \in (0, \dim G/2) \operatorname{sing} \setminus (\dim G)$.*

(i) If $\lfloor \dim G/2 \rfloor$ is even, i.e., if $\dim G \equiv 0 \pmod 4$ or $\dim G \equiv 1 \pmod 4$, then

$$\zeta'((\Gamma \backslash G, \mathbf{m}_\rho), 0) \to -\infty \quad \textit{as} \quad \rho \to \infty \,.$$

(ii) Assume $s \in \left({}^{\dim G}/2 - \mu - 1, {}^{\dim G}/2 - \mu \right)$ *where* $\mu \in \mathbb{N}_0 \cap [0, {}^{\dim G}/2)$. *If* μ *is even, then*

$$\zeta((\Gamma\backslash G, \mathbf{m}_\rho), s) \to -\infty \quad as \quad \rho \to \infty .$$

(iii) (a) $\zeta'((\Gamma\backslash G, \mathbf{m}_\rho), 0) \to +\infty$ *as* $\rho \to 0$. *Consequently, if* $\dim G \equiv 0, 1 \pmod 4$ *then* $[\mathbf{m}] \mapsto \zeta'((\Gamma\backslash G, \mathbf{m}), 0)$ *is neither bounded from above nor below on* $\mathscr{S}\mathscr{M}^{HL}(\Gamma, G)$.

(b) $\zeta((\Gamma\backslash G, \mathbf{m}_\rho), s) \to +\infty$ *as* $\rho \to 0$. *Consequently, if* $s \in \left({}^{\dim G}/2 - \mu - 1, {}^{\dim G}/2 - \mu \right)$ *with* $\mu \in \mathbb{N}_0 \cap [0, {}^{\dim G}/2)$ *and* μ *even, then* $[\mathbf{m}] \mapsto \zeta((\Gamma\backslash G, \mathbf{m}), s)$ *is neither bounded from above nor below on* $\mathscr{S}\mathscr{M}^{HL}(\Gamma, G)$.

PROOF. We use the formulas for $\Gamma(s)\zeta((\Gamma\backslash G, \mathbf{m}), s)$ and $\zeta'((\Gamma\backslash G, \mathbf{m}), 0)$ given in Theorem 2.5 and Corollary 2.6 respectively:

$$\Gamma(s)\zeta((\Gamma\backslash G, \mathbf{m}_\rho), s) = \Gamma(s)\zeta_B((\Gamma\backslash G, \mathbf{m}_\rho), s) + \Gamma(s)\zeta_F((\Gamma\backslash G, \mathbf{m}_\rho), s) , \tag{2.26}$$

$$\zeta'((\Gamma\backslash G, \mathbf{m}_\rho), 0) = \zeta_B'((\Gamma\backslash G, \mathbf{m}_\rho), 0) + \zeta_F'((\Gamma\backslash G, \mathbf{m}_\rho), 0) , \tag{2.27}$$

where $\Gamma(s)\zeta_F((\Gamma\backslash G, \mathbf{m}_\rho), s)$ and $\zeta_F'((\Gamma\backslash G, \mathbf{m}_\rho), 0)$ are given by

$$\begin{aligned} &\frac{\operatorname{Vol} T_{\mathfrak{n},(\mathbf{m}_\rho)_\mathfrak{n}}}{(4\pi)^n} \int_1^\infty \sum_{\lambda \in \Gamma_\mathfrak{z}^*(\rho)\backslash\{0\}} s_{V,t}^{\mathbf{m}_\rho}(\lambda) t^{\tau - n - 1} \mathrm{d}t + \frac{1}{(4\pi)^{\dim G/2}} \int_0^1 \sum_{X \in \Gamma_\mathfrak{z}\backslash\{0\}} \sigma_X^{\mathbf{m}_\rho}(t) t^{\tau - \dim G/2 - 1} \mathrm{d}t \\ &+ \frac{1}{(4\pi)^{\dim G/2}} \Bigg(\int_0^1 \left(\sigma_0^{\mathbf{m}_\rho}(t) - T_{+0}^N[\sigma_0^{\mathbf{m}_\rho}](t) \right) t^{\tau - \dim G/2 - 1} \mathrm{d}t + \sum_{\substack{j=0 \\ j \neq \dim G/2}}^N \frac{a_j^\rho}{\tau - \frac{\dim G}{2} + j} \\ &\qquad + \begin{cases} \gamma \cdot a_{\dim G/2}^\rho & \text{if } \dim G \text{ is even and } \tau = 0 , \\ 0 & \text{if } \dim G \text{ is odd or } \tau > 0 \end{cases} \Bigg) . \end{aligned} \tag{2.28}$$

Here, $N = \lfloor {}^{\dim G}/2 \rfloor$ and $\tau = 0$ in the cases *(i)* and *(iii)(a)*, $N = \mu$ and $\tau = s$ in the cases *(ii)* and *(iii)(b)*, and $T_{+0}^N[\sigma_0^{\mathbf{m}_\rho}](t) = \sum_{j=0}^N a_j^\rho t^j$.

First, we prove *(i)* and *(ii)*. $\Gamma(s)\zeta_B((\Gamma\backslash G, \mathbf{m}_\rho), s)$ and $\zeta_B'((\Gamma\backslash G, \mathbf{m}_\rho), 0)$ both lie in $O(\ln \rho)$ as $\rho \to \infty$ by Lemma 2.17. The two terms in the first line of (2.28) are bounded by Lemma 2.14 and Lemma 2.15. By Lemma 2.18 there exists $C > 0$ such that the integral in the second line of (2.28) is asymptotically equal to

$$(-1)^{N+1} \cdot C \cdot \rho^{({}^1/\ell + {}^1/n)({}^\ell/2 + n - \tau)} \cdot \begin{cases} 1, & \text{if } \dim G \text{ is odd or } \tau > 0 , \\ \ln \rho, & \text{if } \dim G \text{ is even and } \tau = 0 . \end{cases} \tag{2.29}$$

Note that N is even by assumption and (2.29) is thus negative.

We address the sum in the second line of (2.28) including the term on the third line. By Lemma 1.90 each a_j^ρ is a homogeneous polynomial in $c_1^{\mathbf{m}_\rho}, \dots, c_n^{\mathbf{m}_\rho}$ of degree $2j$. Now recall that $c_j^{\mathbf{m}_\rho} = \rho^{1/2\ell+1/2n} \cdot c_j^{\mathbf{m}}$. Hence, $a_j^\rho = \rho^{(1/\ell+1/n)j} \cdot a_j^1$. It follows that there exists a $C_1 \in \mathbb{R}\backslash\{0\}$ such that the sum in the second line including the possibly present term in the third line of (2.28) is asymptotically equal to

$$C_1 \cdot \rho^{(1/\ell+1/n)\cdot N} \tag{2.30}$$

as $\rho \to \infty$. We argue that the order of growth of (2.29) is higher than that of (2.30) in any case:

- Case $\tau = 0$, $\dim G$ odd: Since $\dim G = \ell + 2n$, ℓ is odd. Hence, for $\rho > 1$ we have

$$\begin{aligned}\rho^{(1/\ell+1/n)\cdot N} = \rho^{(1/\ell+1/n)\cdot\lfloor(\ell+2n)/2\rfloor} &= \rho^{(1/\ell+1/n)\cdot(\ell/2-1/2+n)}\\ &< \rho^{(1/\ell+1/n)\cdot(\ell/2+n)}\\ &= \rho^{(1/\ell+1/n)\cdot(\ell/2+n-\tau)}\,.\end{aligned}$$

- Case $\tau = 0$, $\dim G$ even: In this case ℓ is even. Hence,

$$\begin{aligned}\rho^{(1/\ell+1/n)\cdot N} = \rho^{(1/\ell+1/n)\cdot\lfloor(\ell+2n)/2\rfloor} &= \rho^{(1/\ell+1/n)\cdot(\ell/2+n)}\\ &< \rho^{(1/\ell+1/n)\cdot(\ell/2+n)} \cdot \ln\rho\end{aligned}$$

  for $\rho > e$.
- Case $\tau > 0$: This is case *(ii)* and we have $\tau = s \in \left(\dim G/2 - \mu - 1, \dim G/2 - \mu\right)$ and $N = \mu$. Hence,

$$N = \mu < \dim G/2 - \tau = \left(\ell/2 + n\right) - \tau\,,$$

  so that for all $\rho > 1$

$$\rho^{(1/\ell+1/n)\cdot N} < \rho^{(1/\ell+1/n)\cdot(\ell/2+n-\tau)}\,.$$

Clearly, in all three cases (2.29) grows faster than (2.30) and is thus the term which is responsible for the behaviour of (2.28) as $\rho \to \infty$.

It remains to show *(iii)*. We prove the subcases *(a)* and *(b)* together. Lemma 2.17*(ii)* ensures that $\Gamma(s)\zeta_B((\Gamma\backslash G, \mathbf{m}_\rho), s)$ and $\zeta_B'((\Gamma\backslash G, \mathbf{m}_\rho), 0)$ go to $+\infty$ as $\rho \to 0$. By positivity of the first line of (2.28), it suffices to show that the second and third line of (2.28) are bounded. We have seen above that $a_j^\rho = \rho^{(1/\ell+1/n)\cdot j} \cdot a_j^1$ and hence $a_j^\rho \to 0$ as $\rho \to 0$, except for $a_0^\rho \equiv 1$. As for the integral in the second line of (2.28), we have seen in the proof of Lemma 2.18 that $\sigma_0^{\mathbf{m}_\rho}(t) = \sigma_0^{\mathbf{m}}\left(\rho^{1/\ell+1/n}t\right)$. Hence,

$$\int_0^1 \left(\sigma_0^{\mathbf{m}_\rho}(t) - \sum_{j=0}^{N} a_j^\rho t^j\right) t^{\tau-\dim G/2-1}\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^1 \left(\sigma_0^{\mathbf{m}}(\rho^{1/\ell+1/n} \cdot t) - \sum_{j=0}^{N} a_j^1 (\rho^{1/\ell+1/n} \cdot t)^j \right) t^{\tau - \dim G/2 - 1} \mathrm{d}t \\
&= \rho^{(1/\ell+1/n)\cdot(\dim G/2-\tau)} \int_0^{\rho^{1/\ell+1/n}} \left(\sigma_0^{\mathbf{m}}(t) - \sum_{j=0}^{N} a_j^1 t^j \right) t^{\tau - \dim G/2 - 1} \mathrm{d}t \to 0
\end{aligned}
$$

as $\rho \to 0$. □

We now change the setting to Heisenberg manifolds. Here, we introduce a path $\rho \mapsto [f_{n,r}(\rho)]$ in the moduli space of metrics with volume 1. This time, though, the volume of the base and fibre tori will remain constant along it. Just as in the case of the path $\mathbf{m}_\rho$ above, we will prove asymptotics for the various terms in the formulas for the height and ζ-function as $\rho \to \infty$. This will be the contents of Lemma 2.22 - Lemma 2.25. Theorem 2.28 concludes these calculations and is the analog of Theorem 2.19 for $f_{n,r}$.

Recall from Definition 1.70 the moduli space $\mathcal{SM}_n^r$.

DEFINITION 2.20. (cf. Example 1.59) Let $n \geqslant 2$ be an integer and $r \in \mathcal{D}_n$. We define a family of (equivalence classes of) metrics $[f_{n,r}] : [0,\infty) \to \mathcal{SM}_n^r$ by

$$
\begin{aligned}
f_{n,r}(\rho) &:= (h(\rho), g_{n,r}) \,, \\
g_{n,r} &:= |\Gamma^r|^{-2} = \prod_{j=1}^{n} r_j^{-2} \,, \\
h(\rho) &:= \begin{pmatrix} 1 & \rho & \\ \rho & 1+\rho^2 & \\ & & \mathrm{Id}_{2n-2} \end{pmatrix} .
\end{aligned}
$$

REMARK 2.21. Let $\rho \in [0,\infty)$. Then

$$
h(\rho)^{-1} J = \begin{pmatrix} & \begin{matrix} 1+\rho^2 & -\rho \\ -\rho & 1 \end{matrix} & \\ 0 & & \\ & & \mathrm{Id}_{n-2} \\ -\mathrm{Id}_n & 0 & \end{pmatrix}
$$

has square

$$\left(h(\rho)^{-1}J\right)^2 = \begin{pmatrix} -1-\rho^2 & \rho & & & & \\ \rho & -1 & & & & \\ & & -\mathrm{Id}_{n-2} & & & \\ & & & -1-\rho^2 & \rho & \\ & & & \rho & -1 & \\ & & & & & -\mathrm{Id}_{n-2} \end{pmatrix}.$$

Hence the characteristic polynomial $p(X)$ of $\left(h(\rho)^{-1}J\right)^2$ is

$$\begin{aligned} p(X) &= \left(X^2+(2+\rho^2)X+1\right)^2 (X+1)^{2n-4} \\ &= \left(X+\tfrac{\rho^2+2}{2}+\sqrt{\tfrac{(\rho^2+2)^2}{4}-1}\right)^2 \left(X+\tfrac{\rho^2+2}{2}-\sqrt{\tfrac{(\rho^2+2)^2}{4}-1}\right)^2 (X+1)^{2n-4} \\ &= \left(X+\tfrac{\rho^2+2}{2}+\tfrac{\rho}{2}\sqrt{\rho^2+4}\right)^2 \left(X+\tfrac{\rho^2+2}{2}-\tfrac{\rho}{2}\sqrt{\rho^2+4}\right)^2 (X+1)^{2n-4}. \end{aligned}$$

It follows that $d_n(h(\rho)) = \frac{\sqrt{\rho^2+2+\rho\sqrt{\rho^2+4}}}{\sqrt{2}}$, $d_1(h(\rho)) = \frac{\sqrt{\rho^2+2-\rho\sqrt{\rho^2+4}}}{\sqrt{2}}$ and $d_j(h(\rho)) = 1$ for all $j \notin \{1,n\}$ in case $n > 2$. Note that $d_1(h(\rho)) = d_n^{-1}(h(\rho))$ because $\det h(\rho) = 1$.

By Proposition 1.49 we have $c_j^{(h(\rho),g_{n,r})} = g_{n,r}^{1/2} \cdot d_j(h(\rho))$ for all $1 \leqslant j \leqslant n$. Also note that $\mathrm{Vol}(T_{\mathfrak{n},h(\rho)}) = \mathrm{Vol}(\delta_r \cdot \mathbb{Z}^{2n}\backslash\mathbb{R}^{2n}, h(\rho)) = |\Gamma^r| \cdot \det h(\rho)^{1/2} = |\Gamma^r|$ for all $\rho \in [0,\infty)$.

LEMMA 2.22. *Let $n \geqslant 2$ be an integer, $r \in \mathcal{D}_n$ and $s \in [0,\infty)$. Then*

$$(2.31)\quad [0,\infty) \ni \rho \mapsto \zeta_B((\Gamma^r\backslash H_n, f_{n,r}(\rho)),s) = \frac{1}{\Gamma(s)}\Bigg(-\frac{1}{s}+ \\ +\int_1^\infty \sum_{\lambda\in(\Gamma^r_{\mathfrak{n}})^*\backslash\{0\}} e^{-4\pi^2\|\lambda\|^2_{h(\rho)}t} t^{s-1}\mathrm{d}t + \frac{\mathrm{Vol}\,T_{\mathfrak{n},h(\rho)}}{(4\pi)^n}\int_1^\infty \sum_{X\in\Gamma^r_{\mathfrak{n}}\backslash\{0\}} e^{-\frac{\|X\|^2_{h(\rho)}}{4}t} t^{n-s-1}\mathrm{d}t\Bigg) \in \mathbb{R}$$

is a bounded function.

PROOF. The matrix

$$A := \delta_r \begin{pmatrix} 1 & 1 & \\ 0 & 1 & \\ & & \mathrm{Id}_{2n-2} \end{pmatrix} \delta_r^{-1} = \begin{pmatrix} 1 & r_1/r_2 & \\ 0 & 1 & \\ & & \mathrm{Id}_{2n-2} \end{pmatrix}$$

is by definition an element of $G_r = \delta_r\,\mathrm{GL}(2n;\mathbb{Z})\delta_r^{-1}$. Furthermore we have

$${}^{\mathrm{t}}A \cdot h(\rho) \cdot A = \begin{pmatrix} 1 & 0 & \\ r_1/r_2 & 1 & \\ & & \mathrm{Id} \end{pmatrix} \begin{pmatrix} 1 & \rho & \\ \rho & 1+\rho^2 & \\ & & \mathrm{Id}_{2n-2} \end{pmatrix} \begin{pmatrix} 1 & r_1/r_2 & \\ 0 & 1 & \\ & & \mathrm{Id}_{2n-2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \\ {}^{r_1}/_{r_2} & 1 & \\ & & \mathrm{Id} \end{pmatrix} \begin{pmatrix} 1 & \rho + {}^{r_1}/_{r_2} & \\ \rho & 1 + {}^{r_1}/_{r_2}\rho + \rho^2 & \\ & & \mathrm{Id} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \rho + {}^{r_1}/_{r_2} & \\ \rho + {}^{r_1}/_{r_2} & (\rho + {}^{r_1}/_{r_2})^2 + 1 & \\ & & \mathrm{Id} \end{pmatrix}.$$

This means that

$$h(\rho) \equiv h\left(\rho + \tfrac{r_1}{r_2}\right) \mod \delta_r \operatorname{GL}(2n;\mathbb{Z})\delta_r^{-1} \tag{2.32}$$

for all $\rho \in [0,\infty)$

Now note that the value of the function (2.31) only depends on the residue class $[h(\rho)] \in \mathcal{P}_{2n}/G_r$; in fact, the action of any $G \in G_r$ on $h(\rho) \in \mathcal{P}_{2n}$ induces a rearrangement of the terms in each sum in (2.31). By (2.32) the image $[h([0,\infty))] \subset \mathcal{P}_{2n}/G_r$ is compact, which in turn means that the image of the function (2.31) is compact. ☐

LEMMA 2.23. *Let $n \in \mathbb{N}$ with $n \geqslant 2$, $r \in \mathcal{D}_n$ and $s \in [0,\infty)$. Then the function*

$$[0,\infty) \ni \rho \mapsto \frac{\operatorname{Vol} T_{\mathfrak{n},h(\rho)}}{(4\pi)^n} \int_1^\infty \sum_{\lambda \in (\Gamma_{\mathfrak{z}}^r)^* \setminus \{0\}} s_{V,t}^{f_{n,r}(\rho)}(\lambda) t^{s-n-1} \mathrm{d}t$$

$$= \frac{|\Gamma^r|}{(4\pi)^n} \int_1^\infty \sum_{c \in \mathbb{Z}\setminus\{0\}} e^{-4\pi^2 \frac{c^2}{g_{n,r}} t} \prod_{j=1}^n \frac{2\pi c d_j(h(\rho))t}{\sinh(2\pi c d_j(h(\rho))t)} t^{s-n-1} \mathrm{d}t \in \mathbb{R}$$

is bounded.

PROOF. We have

$$\int_1^\infty \sum_{c \in \mathbb{Z}\setminus\{0\}} e^{-4\pi^2 \frac{c^2}{g_{n,r}} t} \prod_{j=1}^n \frac{2\pi c d_j(h(\rho))t}{\sinh(2\pi c d_j(h(\rho))t)} t^{s-n-1} \mathrm{d}t < \int_1^\infty \sum_{c \in \mathbb{Z}\setminus\{0\}} e^{-4\pi^2 \frac{c^2}{g_{n,r}} t} t^{s-n-1} \mathrm{d}t < \infty,$$

where the last inequality holds because of Remark 1.92 and Lemma 2.3. ☐

The following Lemma is the equivalent of Lemma 2.15 for the path $f_{n,r}$. Though the path and the result of the lemma is different, the method of proof is identical.

Recall the definition of $\sigma_X^{\mathbf{m}}$ in (1.43) and its relation to $\widehat{s_{V,t}^{\mathbf{m}}}(X)$ in Theorem 1.78*(iii)*.

LEMMA 2.24. *Let $n \in \mathbb{N}$ with $n \geqslant 2$, $r \in \mathcal{D}_n$ and $s \in [0,\infty)$. Let $\psi_{n,r} : [0,\infty) \to \mathbb{R}$ be the function*

$$\psi_{n,r}(\rho) := \frac{\operatorname{Vol}(\Gamma^r \backslash H_n, (h(\rho), g_{n,r}))}{(4\pi)^{(2n+1)/2}} \int_0^1 \sum_{X \in \Gamma_{\mathfrak{z}}^r \setminus\{0\}} \sigma_X^{(h(\rho), g_{n,r})}(t) t^{s-(2n+1)/2-1} \mathrm{d}t.$$

Then there exist $C, b > 0$ such that for all $\rho > b$ we have

$$0 < \psi_{n,r}(\rho) \leqslant C \begin{cases} 1, & \textit{for } s > n\,, \\ d_n(h(\rho))^{n-s}, & \textit{for } s < n\,, \\ \ln d_n(h(\rho)), & \textit{for } s = n\,. \end{cases}$$

PROOF. First note that $\psi_{n,r} > 0$ by Corollary 1.84. Fix $\rho \in [0, \infty)$ and recall from Remark 2.21 that

$$\begin{aligned} &d_1(h(\rho)) \leqslant 1 = d_2(h(\rho)) = \ldots = d_{n-1}(h(\rho)) \leqslant d_n(h(\rho)) = d_1(h(\rho))^{-1}\,, \\ &\quad d_n(h(\tau)) \to \infty \quad \text{as} \quad \tau \to \infty\,. \end{aligned}$$

In the following we use d_j as short hand for $d_j(h(\rho))$. Let

$$\begin{aligned} \varphi_1(x) &:= e^{-4\pi^2 g_{n,r} x^2}\,, \\ \varphi_2(x) &:= \frac{2\pi g_{n,r} x}{\sinh(2\pi g_{n,r} x)}\,. \end{aligned}$$

Then, by (1.43) and Theorem 1.78*(iii)*:

$$\begin{aligned} \psi_{n,r}(\rho) &= \frac{\mathrm{Vol}(\Gamma^r \backslash H_n, (h(\rho), g_{n,r}))}{(4\pi)^{(2n+1)/2}} \int_0^1 \sum_{X \in \Gamma_{\mathfrak{z}} \backslash \{0\}} \sigma_X^{(h(\rho), g_{n,r})}(t) t^{s-(2n+1)/2-1} \mathrm{d}t \\ &= \frac{\mathrm{Vol}(\Gamma^r \backslash H_n, (h(\rho), g_{n,r}))}{(4\pi)^n} \int_0^1 \sum_{X \in \Gamma_{\mathfrak{z}} \backslash \{0\}} s_{V,t}^{\widehat{(h(\rho), g_{n,r})}}(X) t^{s-n-1} \mathrm{d}t \\ &= \frac{1}{(4\pi)^n} \int_0^1 \sum_{c \in \mathbb{Z} \backslash \{0\}} \int_{\mathbb{R}} e^{-2\pi i g_{n,r} c \cdot \xi} \varphi_1\left(t^{1/2}\xi\right) \prod_{j=1}^n \varphi_2(d_j t\xi) g_{n,r}^{1/2} \mathrm{d}\xi t^{s-n-1} \mathrm{d}t \\ &= \frac{1}{(4\pi)^n} \int_0^{d_n} \sum_{c \in \mathbb{Z} \backslash \{0\}} \int_{\mathbb{R}} e^{-2\pi i g_{n,r} c \cdot \xi} \varphi_1\left(t^{1/2} d_n^{-1/2} \xi\right) \varphi_2(d_n^{-2} t\xi) \\ &\quad \cdot \varphi_2(d_n^{-1} t\xi)^{n-2} \varphi_2(t\xi) g_{n,r}^{1/2} \mathrm{d}\xi \left(d_n^{-1} t\right)^{s-n-1} d_n^{-1}\, \mathrm{d}t \end{aligned}$$

$$\begin{aligned} &= d_n^{n-s} \frac{1}{(4\pi)^n} \int_0^{d_n} \sum_{c \in \mathbb{Z} \backslash \{0\}} \int_{\mathbb{R}} e^{-2\pi i g_{n,r} c \cdot \xi} \varphi_1\left(t^{1/2} d_n^{-1/2} \xi\right) \varphi_2(d_n^{-2} t\xi) \\ &\quad \cdot \varphi_2(d_n^{-1} t\xi)^{n-2} \varphi_2(t\xi) g_{n,r}^{1/2} \mathrm{d}\xi t^{s-n-1} \mathrm{d}t\,. \end{aligned} \tag{2.33}$$

We split the outer integral in (2.33) into $\int_0^{d_n} = \int_0^1 + \int_1^{d_n}$ and treat $\int_0^1$ first.

We will show that $\lim_{d_n \to \infty} \int_0^1$ converges to a real number. To compute this limit we have to cross 3 borders: an integral, a sum and another integral. The innermost integral poses no problem: Since $0 \leqslant \varphi_{1,2} \leqslant 1$ we have for all $t > 0$ and all d_n:

$$\left| e^{-2\pi i g_{n,r} c\cdot\xi} \varphi_1\left(t^{1/2} d_n^{-1/2}\xi\right) \varphi_2(d_n^{-2}t\xi)\varphi_2(d_n^{-1}t\xi)^{n-2}\varphi_2(t\xi)\right| \leqslant \varphi_2(t\xi)\,.$$

It follows via Theorem 1.36 and Remark 1.77 that

$$\lim_{d_n\to\infty}\int_{\mathbb{R}} e^{-2\pi i g_{n,r} c\cdot\xi} \varphi_1\left(t^{1/2} d_n^{-1/2}\xi\right) \varphi_2(d_n^{-2}t\xi)\varphi_2(d_n^{-1}t\xi)^{n-2}\varphi_2(t\xi) g_{n,r}^{1/2}\mathrm{d}\xi$$
$$= \int_{\mathbb{R}} e^{-2\pi i g_{n,r} c\cdot\xi}\varphi_2(t\xi) g_{n,r}^{1/2}\mathrm{d}\xi = \frac{\pi}{4t g_{n,r}^{1/2}}\operatorname{sech}^2\left(\frac{\pi c}{2t}\right)\,. \tag{2.34}$$

To be able to exchange the limit with the remaining series and integral, we first have to prove an estimate for the innermost integral. We denote that integral by $I(c,t,d_n)$, i.e.,

$$I(c,t,d_n) := \int_{\mathbb{R}} e^{-2\pi i g_{n,r} c\cdot\xi} \varphi_1\left(t^{1/2} d_n^{-1/2}\xi\right) \varphi_2(d_n^{-2}t\xi)\varphi_2(d_n^{-1}t\xi)^{n-2}\varphi_2(t\xi) g_{n,r}^{1/2}\mathrm{d}\xi\,.$$

LEMMA 2.25. *For all $N \in \mathbb{N}$ there exists $C > 0$ such that*

$$|I(c,t,d_n)| \leqslant C\cdot\frac{t^N}{|c|^{2(N+1)}}$$

for all $c \in \mathbb{Z}\backslash\{0\}$, $t > 0$ and $d_n \in [1,\infty)$.

PROOF. By repeated integration by parts we have for every $k \in \mathbb{N}$

$$(-2\pi i g_{n,r}c)^k\cdot I(c,t,d_n)$$
$$= (-1)^k\int_{\mathbb{R}} e^{-2\pi i g_{n,r} c\cdot\xi}\frac{\partial^k}{\partial\xi^k}\left(\varphi_1\left(t^{1/2} d_n^{-1/2}\xi\right) \varphi_2(d_n^{-2}t\xi)\varphi_2(d_n^{-1}t\xi)^{n-2}\varphi_2(t\xi)\right) g_{n,r}^{1/2}\mathrm{d}\xi\,.$$

Hence,

$$|2\pi i g_{n,r}c|^k\cdot|I(c,t,d_n)|$$
$$\leqslant \int_{\mathbb{R}}\left|\frac{\partial^k}{\partial\xi^k}\left(\varphi_1\left(t^{1/2} d_n^{-1/2}\xi\right) \varphi_2(d_n^{-2}t\xi)\varphi_2(d_n^{-1}t\xi)^{n-2}\varphi_2(t\xi)\right)\right| g_{n,r}^{1/2}\mathrm{d}\xi\,. \tag{2.35}$$

Now note that $\frac{\partial^k}{\partial\xi^k}\left(\varphi_1\left(t^{1/2} d_n^{-1/2}\xi\right) \varphi_2(d_n^{-2}t\xi)\varphi_2(d_n^{-1}t\xi)^{n-2}\varphi_2(t\xi)\right)$ is a linear combination of functions of the form

$$\mathbb{R}\ni\xi\mapsto t^{\alpha/2}d_n^{-\beta/2}\psi_1\left(t^{1/2} d_n^{-1/2}\xi\right)\psi_2(d_n^{-2}t\xi)\psi_3(d_n^{-1}t\xi)\psi_4(t\xi)\in\mathbb{R}\,, \tag{2.36}$$

where $\psi_j \in \mathscr{S}(\mathbb{R})$, $\alpha,\beta\in\mathbb{N}_0$ with $\alpha \geqslant k$. Hence, there exists $C' > 0$ such that

$$t^{\alpha/2}d_n^{-\beta/2}\int_{\mathbb{R}}\left|\psi_1\left(t^{1/2} d_n^{-1/2}\xi\right)\psi_2(d_n^{-2}t\xi)\psi_3(d_n^{-1}t\xi)\psi_4(t\xi)\right|\mathrm{d}\xi$$

$$= t^{\alpha/2-1} d_n^{-\beta/2} \int_{\mathbb{R}} \left| \psi_1 \left(t^{-1/2} d_n^{-1/2} \xi \right) \psi_2(d_n^{-2}\xi) \psi_3(d_n^{-1}\xi) \psi_4(\xi) \right| \mathrm{d}\xi$$

$$\leqslant t^{\alpha/2-1} d_n^{-\beta/2} \int_{\mathbb{R}} \max |\psi_1\psi_2\psi_3| \cdot |\psi_4(\xi)| \, \mathrm{d}\xi < t^{\alpha/2-1} d_n^{-\beta/2} \cdot C'$$

for all $t > 0$ and all $d_n \in [1, \infty)$. From this and (2.35) it follows that for every $k \in \mathbb{N}$ there exists $C > 0$ such that

$$|c|^k \cdot |I(c,t,d_n)| \leqslant C \cdot t^{k/2-1} \tag{2.37}$$

for all $c \in \mathbb{Z}\backslash\{0\}$, $t \in (0,1]$ and $d_n \in [1,\infty)$. Let now $N \in \mathbb{N}$. With the choice $k = 2(N+1)$ we obtain the desired result from (2.37). □

We return to the proof of

$$\lim_{d_n\to\infty} \int_0^1 \sum_{c\in\mathbb{Z}\backslash\{0\}} I(c,t,d_n) \cdot t^{s-n-1} \mathrm{d}t = \int_0^1 \sum_{c\in\mathbb{Z}\backslash\{0\}} \lim_{d_n\to\infty} I(c,t,d_n) \cdot t^{s-n-1} \mathrm{d}t < \infty .$$

Choose $N \in \mathbb{N}$ with $N > n - s$. By the last lemma there exists $C' > 0$ such that $|I(c,t,d_n)| \leqslant C' \cdot t^N / |c|^{2(N+1)}$ for every $c \in \mathbb{Z}\backslash\{0\}$, $t \in (0,1]$ and $d_n \in [1,\infty)$. Hence,

$$\sum_{c\in\mathbb{Z}\backslash\{0\}} \sup_{d_n\in[1,\infty)} |I(c,t,d_n)| \leqslant C' \cdot t^N \cdot 2 \cdot \zeta_R(2(N+1)) ,$$

where ζ_R is the Riemann ζ-function. It follows that $\sum_c I(c,t,d_n)$ converges normally on $(t, d_n) \in (0,1] \times [1,\infty)$ and that

$$\left| \sum_{c\in\mathbb{Z}\backslash\{0\}} I(c,t,d_n) \right| \cdot t^{s-n-1} \leqslant C' \cdot 2 \cdot \zeta_R(2(N+1)) \cdot t^{N+s-n-1} .$$

The right hand side is integrable on $t \in (0,1]$ by choice of N. With Theorem 1.36 and (2.34) we finally arrive at the claim

$$\begin{aligned} \lim_{d_n\to\infty} \int_0^1 \sum_{c\in\mathbb{Z}\backslash\{0\}} I(c,t,d_n) \cdot t^{s-n-1} \mathrm{d}t &= \int_0^1 \lim_{d_n\to\infty} \sum_{c\in\mathbb{Z}\backslash\{0\}} I(c,t,d_n) \cdot t^{s-n-1} \mathrm{d}t \\ &= \int_0^1 \sum_{c\in\mathbb{Z}\backslash\{0\}} \lim_{d_n\to\infty} I(c,t,d_n) \cdot t^{s-n-1} \mathrm{d}t \\ &= \frac{\pi}{4 g_{n,r}^{1/2}} \int_0^1 \sum_{c\in\mathbb{Z}\backslash\{0\}} \operatorname{sech}^2\left(\frac{\pi c}{2t}\right) \cdot t^{s-n-2} \mathrm{d}t < \infty . \end{aligned} \tag{2.38}$$

We will now calculate bounds for the integral $\int_1^{d_n}$ in (2.33) as $d_n \to \infty$. Note that φ_1, φ_2 are even functions and that the Fourier transform of such is its own inverse. With an application of Poisson's summation formula (see Corollary 1.35) we calculate

$$\begin{aligned}
(2.39)\quad & \int_1^{d_n} \sum_{c\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} e^{-2\pi i g_{n,r} c\cdot\xi} \varphi_1\left(t^{1/2} d_n^{-1/2}\xi\right) \varphi_2(d_n^{-2}t\xi)\varphi_2(d_n^{-1}t\xi)^{n-2}\varphi_2(t\xi) g_{n,r}^{1/2} \mathrm{d}\xi\, t^{s-n-1}\mathrm{d}t \\
&= \int_1^{d_n} \sum_{c\in\mathbb{Z}} \int_{\mathbb{R}} e^{-2\pi i g_{n,r} c\cdot\xi} \varphi_1\left(t^{1/2} d_n^{-1/2}\xi\right) \varphi_2(d_n^{-2}t\xi)\varphi_2(d_n^{-1}t\xi)^{n-2}\varphi_2(t\xi) g_{n,r}^{1/2} \mathrm{d}\xi\, t^{s-n-1}\mathrm{d}t \\
&\qquad - \int_1^{d_n}\int_{\mathbb{R}} \varphi_1\left(t^{1/2} d_n^{-1/2}\xi\right) \varphi_2(d_n^{-2}t\xi)\varphi_2(d_n^{-1}t\xi)^{n-2}\varphi_2(t\xi) g_{n,r}^{1/2} \mathrm{d}\xi\, t^{s-n-1}\mathrm{d}t \\
&= g_{n,r}^{-1/2} \int_1^{d_n} \sum_{c\in\mathbb{Z}\setminus\{0\}} \varphi_1\left(t^{1/2} d_n^{-1/2} g_{n,r}^{-1} c\right) \varphi_2(d_n^{-2}t g_{n,r}^{-1} c)\varphi_2(d_n^{-1}tc)^{n-2}\varphi_2(t g_{n,r}^{-1} c) t^{n-s-1}\mathrm{d}t \\
&\qquad\qquad + g_{n,r}^{-1/2} \int_1^{d_n} t^{s-n-1}\mathrm{d}t \\
&\qquad - \int_1^{d_n}\int_{\mathbb{R}} \varphi_1\left(t^{1/2} d_n^{-1/2}\xi\right) \varphi_2(d_n^{-2}t\xi)\varphi_2(d_n^{-1}t\xi)^{n-2}\varphi_2(t\xi) g_{n,r}^{1/2} \mathrm{d}\xi\, t^{s-n-1}\mathrm{d}t \\
&\qquad \leqslant g_{n,r}^{-1/2} \int_1^{\infty} \sum_{c\in\mathbb{Z}\setminus\{0\}} \varphi_2(t g_{n,r}^{-1} c) t^{s-n-1}\mathrm{d}t + g_{n,r}^{-1/2} \begin{cases} \frac{1}{s-n}\left(d_n^{s-n} - 1\right), & \text{for } s \neq n\,, \\ \log d_n, & \text{for } s = n\,, \end{cases}
\end{aligned}$$

where we have simply dropped the last term due to its nonpositivity and bounded the integrand with $\varphi_1 \cdot \varphi_2 \cdot \varphi_2^{n-2} \cdot \varphi_2 \leqslant \varphi_2$. The integral on the last line is finite by Remark 1.92 and Lemma 2.3. Putting (2.33), (2.38) and (2.39) together, we conclude that there exist $C, b > 0$ such that

$$\psi_{n,r}(\rho) \leqslant C \begin{cases} 1, & \text{for } s > n\,, \\ d_n(h(\rho))^{n-s}, & \text{for } s < n\,, \\ \ln d_n(h(\rho)), & \text{for } s = n\,, \end{cases}$$

for all $\rho > b$. □

The following lemma is the equivalent of Lemma 2.18 for the path $f_{n,r}(\rho)$. Unfortunately, we have to work harder this time.

LEMMA 2.26. *Let $n \in \mathbb{N}$, $r \in \mathcal{D}_n$ and $s \in [0, (2n+1)/2) \setminus \operatorname{sing}(2n+1)$. Let $N \in \mathbb{N}_0$ be the smallest number such that $N > (2n+1)/2 - s - 1$. Then there exists $C > 0$ such that*

$$\int_0^1 \left(\sigma_0^{f_{n,r}(\rho)}(t) - T_{+0}^N[\sigma_0^{f_{n,r}(\rho)}](t)\right) t^{s-\frac{2n+1}{2}-1}\mathrm{d}t \sim (-1)^{N+1} \cdot C \cdot d_n(h(\rho))^{2n+1-2s} \tag{2.40}$$

as $\rho \to \infty$.

PROOF. We first note that by Corollary 1.89 the integrand on the left hand side is nonzero and has sign $(-1)^{N+1}$ for all $t \in (0,1]$. Let $\varphi(x) = \sqrt{g_{n,r}}x \operatorname{csch}(\sqrt{g_{n,r}}x)$ and abbreviate $d_j(h(\rho))$ to d_j. Furthermore, let $T_{+0}^N[\sigma_0^{f_{n,r}(\rho)}](t) = \sum_{j=0}^N a_j^\rho t^j$. Then, using the expression for $\sigma_0^{f_{n,r}(\rho)}$ given in Remark 1.80 and recalling that $c_j^{f_{n,r}(\rho)} = \sqrt{g_{n,r}}d_j(h(\rho))$, we have

$$\begin{aligned}
&\int_0^1 \left(\sigma_0^{f_{n,r}(\rho)}(t) - T_{+0}^N[\sigma_0^{f_{n,r}(\rho)}](t)\right) t^{s-\frac{2n+1}{2}-1}\mathrm{d}t \\
=&\int_0^1 \left(\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-r^2}\varphi(\sqrt{t}rd_1)\varphi(\sqrt{t}rd_n)\varphi(\sqrt{t}r)^{n-2}\mathrm{d}r - \sum_{j=0}^N a_j^\rho t^j\right) t^{s-\frac{2n+1}{2}-1}\mathrm{d}t \\
=&d_n^{2n+1-2s}\int_0^{d_n^2} \left(\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-r^2}\varphi(\sqrt{t}rd_n^{-2})\varphi(\sqrt{t}r)\varphi(\sqrt{t}rd_n^{-1})^{n-2}\mathrm{d}r - \sum_{j=0}^N a_j^\rho \frac{t^j}{d_n^{2j}}\right) t^{s-\frac{2n+1}{2}-1}\mathrm{d}t \\
=&d_n^{2n+1-2s}\int_0^{d_n^2} \left(F_\rho(t) - T_{+0}^N[F_\rho](t)\right) t^{s-\frac{2n+1}{2}-1}\mathrm{d}t\,,
\end{aligned}$$

where

$$f_\rho(t,r) := \tfrac{2}{\sqrt{\pi}}e^{-r^2}\varphi(\sqrt{t}rd_n^{-2}(h(\rho)))\varphi(\sqrt{t}r)\varphi(\sqrt{t}rd_n^{-1}(h(\rho)))^{n-2}\,, \quad F_\rho(t) := \int_0^\infty f_\rho(t,r)\mathrm{d}r\,.$$

Define also

$$f(t,r) := \tfrac{2}{\sqrt{\pi}}e^{-r^2}\varphi(\sqrt{t}r)\,, \qquad F(t) := \int_0^\infty f(t,r)\mathrm{d}r\,.$$

We claim that

$$\int_0^{d_n^2} \left(F_\rho(t) - T_{+0}^N[F_\rho](t)\right) t^{s-\frac{2n+1}{2}-1}\mathrm{d}t \to \int_0^\infty \left(F(t) - T_{+0}^N[F](t)\right) t^{s-\frac{2n+1}{2}-1}\mathrm{d}t \tag{2.41}$$

as $\rho \to \infty$ and that the right hand side is finite and has sign $(-1)^{N+1}$.

LEMMA 2.27. *(i) For all $j \in \mathbb{N}$: $\frac{\mathrm{d}^j}{\mathrm{d}t^j} F_\rho(t) = \int_0^\infty \frac{\partial^j}{\partial t^j} f_\rho(t,r)\mathrm{d}r$ and $\frac{\mathrm{d}^j}{\mathrm{d}t^j} F(t) = \int_0^\infty \frac{\partial^j}{\partial t^j} f(t,r)\mathrm{d}r$.*

(ii) For all $j \in \mathbb{N}$: $\frac{\mathrm{d}^j}{\mathrm{d}t^j} F_\rho(t)$ converges uniformly to $\frac{\mathrm{d}^j}{\mathrm{d}t^j} F(t)$ on $[0,\infty)$ as $\rho \to \infty$.

PROOF. Note that there is a function $\psi : (-\epsilon,\infty) \to \mathbb{R}$ with $\varphi(\sqrt{x}) = \psi(x)$ for all $x \in [0,\infty)$ such that $\psi_{|[0,\infty)}$ is completely monotone (see Lemma 1.75 and the related Remark 1.87). In particular,

$$M_j := \max_{\ell \in \{0,\dots,j\}} \sup\{\psi^{(\ell)}(x) \mid x \geqslant 0\} < \infty$$

by Remark 1.73. Moreover, we have

$$\frac{\partial^j}{\partial t^j} f_\rho(t,r) = e^{-r^2} \cdot \sum_a \tau_a(t,r,\rho)\,,$$

where the sum is finite and each $\tau_a(t,r,\rho)$ is the product of a monomial $\mathrm{mon}_a(r, d_n^{-1}(h(\rho))) = c_a \cdot r^{\mu_a} \cdot d_n^{-\nu_a}(h(\rho))$ with $c_a \in \mathbb{R}; \mu_a, \nu_a \in \mathbb{N}_0$ and the product of up to j-th derivatives of ψ at one of the points tr^2, $tr^2 d_n^{-4}(h(\rho))$, $tr^2 d_n^{-2}(h(\rho))$.

Define

$$g_j(r) := e^{-r^2} \cdot M_j^n \cdot \sum_a |\mathrm{mon}_a(r,1)|\;.$$

Since $d_n(h(\rho)) \geqslant 1$ for all $\rho \geqslant 0$ we have for all $\rho \in [0,\infty)$:

$$\left|\frac{\partial^j}{\partial t^j} f_\rho(t,r)\right| \leqslant g_j(r) \quad \text{for all } t, r \in [0,\infty)\,. \tag{2.42}$$

From (2.42) it follows iteratively by Theorem 1.37 *(Supplement)* that

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j} F_\rho(t) = \int_0^\infty \frac{\partial^j}{\partial t^j} f_\rho(t,r)\mathrm{d}r\,.$$

One shows analogously that

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j} F(t) = \int_0^\infty \frac{\partial^j}{\partial t^j} f(t,r)\mathrm{d}r\,.$$

Now note that

$$\left|\frac{\partial^j}{\partial t^j} f_\rho(t,r) - \frac{\partial^j}{\partial t^j} f(t,r)\right| \leqslant e^{-r^2} \cdot M_j^n \cdot \sum_a \left|\mathrm{mon}_a(r, d_n^{-1}(h(\rho))) - \mathrm{mon}_a(r,0)\right|\,.$$

Define $p_{j,\rho}(r)$ to be the sum on the right hand side. Then we have by *(i)*:

$$\left|\frac{\mathrm{d}^j}{\mathrm{d}t^j} F_\rho(t) - \frac{\mathrm{d}^j}{\mathrm{d}t^j} F(t)\right| \leqslant M_j^n \cdot \int_0^\infty e^{-r^2} \cdot p_{j,\rho}(r)\mathrm{d}r \to 0$$

as $\rho \to \infty$, which proves (ii). ☐

We return to the proof of Lemma 2.26. Lemma 2.27*(ii)* in $t = +0$ reads

$$j! \cdot \frac{a_j^\rho}{d_n^{2j}(h(\rho))} \to j! \cdot b_j := \frac{\mathrm{d}^j}{\mathrm{d}t^j}_{|t=0} F(t)$$

as $\rho \to \infty$. In particular, we have

$$T_{+0}^N[F_\rho](t) = \sum_{j=0}^N a_j^\rho \tfrac{t^j}{d_n^{2j}} \to \sum_{j=0}^N b_j t^j = T_{+0}^N[F](t)$$

as $\rho \to \infty$.

$\big(F(t) - T_{+0}^N[F](t)\big) \cdot t^{s-(2n+1)/2-1}$ is integrable on $(0,\infty)$: By Taylor's Theorem with Lagrange remainder term we have $|F(t) - T_{+0}^N[F](t)| \leqslant \frac{1}{(N+1)!} \cdot \mu \cdot t^{N+1}$ for all $t \in [0,1]$, for some $\mu > 0$. Hence

$$\int_0^1 \Big|F(t) - T_{+0}^N[F](t)\Big| \cdot t^{s-(2n+1)/2-1} \mathrm{d}t \leqslant \frac{1}{(N+1)!} \cdot \mu \cdot \int_0^1 t^{N+s-(2n+1)/2} \mathrm{d}t < \infty \tag{2.43}$$

by choice of N. On the other hand, $F(t) \leqslant \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-r^2} \mathrm{d}r = 1$ and thus

$$\int_1^\infty F(t) \cdot t^{s-(2n+1)/2-1} \mathrm{d}t \leqslant \int_1^\infty t^{s-(2n+1)/2-1} \mathrm{d}t < \infty \tag{2.44}$$

as well as

$$\int_1^\infty |b_j| \cdot t^j \cdot t^{s-(2n+1)/2-1} \mathrm{d}t = |b_j| \cdot \int_1^\infty t^{j+s-(2n+1)/2-1} \mathrm{d}t < \infty \text{ for } j \in \{0, 1, \ldots, N\}.$$

Analogously, one shows that $\big(F_\rho(t) - T_{+0}^N[F_\rho](t)\big) \cdot t^{s-(2n+1)/2-1}$ is integrable on $(0,\infty)$.

To show (2.41) it now suffices to prove that

$$\lim_{\rho\to\infty} \int_0^\infty \Big|(F_\rho - F)(t) - T_{+0}^N[F_\rho - F](t)\Big| \cdot t^{s-(2n+1)/2-1} \mathrm{d}t = 0 \tag{2.45}$$

since then we also have

$$\lim_{\rho\to\infty} \int_0^{d_n^2(h(\rho))} \Big|(F_\rho - F)(t) - T_{+0}^N[F_\rho - F](t)\Big| \cdot t^{s-(2n+1)/2-1} \mathrm{d}t = 0\,.$$

To prove (2.45) we first note that $|(F_\rho - F)(t)| \cdot t^{s-(2n+1)/2-1} \to 0$ for every $t \in [1,\infty)$ as $\rho \to \infty$ and that

$$|(F_\rho - F)(t)| \cdot t^{s-(2n+1)/2-1} \leqslant 2F(t) \cdot t^{s-(2n+1)/2-1}\,.$$

We have already proven that the right hand side is integrable, see (2.44). It follows from the Dominated Convergence Theorem that

$$\int_1^\infty |(F_\rho - F)(t)| \cdot t^{s-(2n+1)/2-1} \mathrm{d}t \to 0 \quad \text{as} \quad \rho \to \infty\,.$$

We also have

$$\int_1^\infty \left| T_{+0}^N [F_\rho - F](t) \right| \cdot t^{s-(2n+1)/2-1} \leqslant \max_{j\in\{0,\dots,N\}} \left| \frac{a_j^\rho}{d_n^{2j}(h(\rho))} - b_j \right| \cdot \sum_{j=0}^N \int_1^\infty t^{j+s-(2n+1)/2-1} \mathrm{d}t \to 0$$

as $\rho \to \infty$.

Let $G_\rho := F_\rho - F$. It remains to show that

$$\int_0^1 \left| G_\rho(t) - T_{+0}^N [G_\rho](t) \right| \cdot t^{s-(2n+1)/2-1} \mathrm{d}t \to 0 \quad \text{as} \quad \rho \to \infty\,.$$

By Taylor's Theorem with Lagrange remainder, we have

$$\left| G_\rho(t) - T_{+0}^N [G_\rho](t) \right| \leqslant \tfrac{1}{N+1} \cdot G_\rho^{(N+1)}(\xi_t) t^{N+1}$$

for some $\xi_t \in (0,1)$. By Lemma 2.27*(ii)* we have $\max_{t\in[0,1]} |G_\rho^{(N+1)}(t)| \to 0$ as $\rho \to \infty$. Hence,

$$\int_0^1 \left| G_\rho(t) - T_{+0}^N [G_\rho](t) \right| \cdot t^{s-(2n+1)/2-1} \mathrm{d}t \leqslant \tfrac{1}{(N+1)!} \max_{t\in[0,1]} |G_\rho^{(N+1)}(t)| \int_0^1 t^{N+s-(2n+1)/2} \mathrm{d}t \to 0$$

as $\rho \to \infty$.

Note that we still have

$$(-1)^{N+1} \left(F(t) - \sum_{j=0}^N b_j t^j \right) \geqslant 0$$

for all $t \geqslant 0$ with equality if and only if $t = 0$. Indeed, this is the statement of Corollary 1.89 with $n = 1$ and $\mathbf{m}$ any metric with $c_1^{\mathbf{m}} = \sqrt{g_{n,r}}$. Hence, by defining

$$C := \left| \int_0^\infty \Big(F(t) - T_{+0}^N [F](t) \Big) \cdot t^{s-(2n+1)/2-1} \mathrm{d}t \right|\,,$$

we finally arrive at the desired result. □

THEOREM 2.28. *Let $n \in \mathbb{N}$ with $n \geqslant 2$, $r \in \mathcal{D}_n$ and $s \in (0,\infty)\backslash \operatorname{sing}(2n+1)$.*

(i) If n is even, i.e., $\dim(\Gamma^r \backslash H_n) \equiv 1 \pmod 4$, then $\zeta'((\Gamma^r\backslash H_n, f_{n,r}(\rho)), 0) \to -\infty$ as $\rho \to \infty$. By this and Theorem 2.10, it follows that the height $[\mathbf{m}] \mapsto \zeta'_H((\Gamma^r\backslash H_n, \mathbf{m}), 0)$ is neither bounded from above nor below on $\mathcal{SM}_n^r$.

(ii) Assume $s \in \left(\frac{2n+1}{2} - \mu - 1, \frac{2n+1}{2} - \mu\right)$ *where* $\mu \in \{0, 1, \ldots, n\}$. *If* μ *is even, then* $\zeta((\Gamma^r \backslash H_n, f_{n,r}(\rho)), s) \to -\infty$ *as* $\rho \to \infty$. *By this and Theorem 2.10, it follows that* $[\mathbf{m}] \mapsto \zeta((\Gamma^r \backslash H_n, \mathbf{m}), s)$ *is neither bounded from above nor below on* $\mathcal{SM}_n^r$.

PROOF. We use the representation of $\Gamma(s)\zeta((\Gamma^r \backslash H_n, f_{n,r}(\rho)), s)$ and $\zeta'((\Gamma^r \backslash H_n, f_{n,r}(\rho)), 0)$ given in Theorem 2.5 and Corollary 2.6 and have a look at how the different terms behave along the path $f_{n,r}$:

(2.46)
$$\begin{aligned} &C(\tau) + \int_1^\infty \sum_{\lambda \in (\Gamma_{\mathfrak{n}}^r)^* \backslash \{0\}} e^{-4\pi^2 \|X\|^2_{h(\rho)} t} t^{\tau - 1} \mathrm{d}t + \frac{\operatorname{Vol} T_{\mathfrak{n}, h(\rho)}}{(4\pi)^n} \int_1^\infty \sum_{X \in \Gamma_{\mathfrak{n}}^r \backslash \{0\}} e^{-\frac{\|X\|^2_{h(\rho)}}{4} t} t^{n - \tau - 1} \mathrm{d}t \\ &+ \frac{\operatorname{Vol} T_{\mathfrak{n}, h(\rho)}}{(4\pi)^n} \int_1^\infty \sum_{\lambda \in (\Gamma_{\mathfrak{z}}^r)^* \backslash \{0\}} s_{V,t}^{f_{n,r}(\rho)}(\lambda) t^{\tau - n - 1} \mathrm{d}t + \frac{1}{(4\pi)^{(2n+1)/2}} \int_0^1 \sum_{X \in \Gamma_{\mathfrak{z}} \backslash \{0\}} \sigma_X^{f_{n,r}(\rho)} t^{\tau - (2n+1)/2 - 1} \mathrm{d}t \\ &+ \frac{1}{(4\pi)^{(2n+1)/2}} \left(\int_0^1 \left(\sigma_0^{f_{n,f}(\rho)}(t) - T^N_{+0}[\sigma_0^{f_{n,f}(\rho)}](t) \right) t^{\tau - (2n+1)/2 - 1} \mathrm{d}t + \sum_{j=0}^N \frac{a_j^\rho}{\tau - (2n+1)/2 + j} \right), \end{aligned}$$

where $C(\tau) \in \mathbb{R}$ is a constant, $T^N_{+0}[\sigma_0^{f_{n,r}(\rho)}](t) = \sum_{j=0}^N a_j^\rho t^j$, $\tau = 0$ in case *(i)*, $\tau = s$ in case *(ii)* and $N := n$ in case *(i)* and $N := \mu$ in case *(ii)*.

The first two nonconstant terms of (2.46) are bounded as $\rho \to \infty$ by Lemma 2.22. The first term on the second line is bounded as $\rho \to \infty$ by Lemma 2.23.

Now we address the unbounded terms. By Lemma 2.24 there exists a $C_0 > 0$ such that the second term in the second line is bounded by

$$C_0 \cdot \begin{cases} 1, & \text{for } \tau > n\,, \\ d_n(h(\rho))^{n-\tau}, & \text{for } \tau < n\,, \\ \ln d_n(h(\rho)), & \text{for } \tau = n\,. \end{cases} \tag{2.47}$$

Lemma 2.26 tells us that there exists a $C_1 > 0$ such that the first term on the third line of (2.46) is asymptotically equal to

$$-C_1 \cdot d_n(h(\rho))^{2n+1-2\tau}\,. \tag{2.48}$$

We have a look at the sum in the third line of (2.46). Recalling that $c_j^{f_{n,r}(\rho)} = \sqrt{g_{n,r}} d_j(h(\rho))$ and using Lemma 1.90 we see that each a_j^ρ is a homogeneous polynomial in $d_1(h(\rho)), \ldots, d_n(h(\rho))$ of degree $2j$ and the coefficient of each monomial has sign $(-1)^j$. Hence, the term $\frac{a_N^\rho}{\tau - (2n+1)/2 + N}$ contains the highest degree (=$2N$) monomials, in particular the term $d_n(h(\rho))^{2N}$. Since $\tau - (2n+1)/2 + N < 0$, there exists $C_2 > 0$ such that the sum on the third line is asymptotically equal to

$$-C_2 \cdot d_n(h(\rho))^{2N}\,. \tag{2.49}$$

Since $n+1 > {(2n+1)}/{2} > \tau$ we have $2n+1-2\tau = (n+1)+(n-2\tau) > \tau+(n-2\tau) = n-\tau$. It follows that the term (2.48) has higher order than the term (2.47) as $\rho \to \infty$. Therefore, (2.46) goes to $-\infty$ as $\rho \to \infty$.

Lastly, we note that $\zeta'((\Gamma^r\backslash H_n, \mathbf{m}), 0) \to \infty$ and $\zeta((\Gamma^r\backslash H_n, \mathbf{m}), s) \to \infty$ as $[\mathbf{m}] \to \partial\mathcal{SM}_n^{r,HT}$ by Theorem 2.10, respectively, and that both functions are thus neither bounded from above nor below under the stated conditions. □

REMARK 2.29. Let G be a connected and simply connected nonsingular 2-step nilpotent Lie group with $\dim G = \ell + 2n$, where ℓ is the dimension of the centre of G. Let $\Gamma \subset G$ a uniform subgroup and assume that there exists $[\mathbf{m}] \in \mathscr{SM}^{HL}(\Gamma, G)$. Until now we have investigated the existence of global bounds/extrema for $[\mathbf{m}] \mapsto \zeta((\Gamma\backslash G, \mathbf{m}), s)$ only for $s \in (0, {\dim G}/{2})$. However, the case $s > {\dim G}/{2}$ is not very hard. Let $\mathbf{m}_\rho$ be the metric from Proposition 2.12 and $s \in ({\dim G}/{2}, \infty)$. Denote by $\Gamma_{\mathfrak{z}}^*(\rho)$ and $\Gamma_{\mathfrak{n}}^*(\rho)$ the $\mathbf{m}_\rho$-dual lattices to $\Gamma_{\mathfrak{z}}$ and $\Gamma_{\mathfrak{n}}$ respectively. We represent $\zeta((\Gamma\backslash G, \mathbf{m}_\rho), s)$ by its defining Dirichlet series, which is possible since this converges for $\Re s > {\dim G}/{2}$. By Corollary 1.14 and Proposition 2.12 we have

$$\begin{aligned}
\zeta((\Gamma\backslash G, \mathbf{m}_\rho), s) &= \sum_{\lambda \in \Gamma_{\mathfrak{n}}^*(\rho)\backslash\{0\}} \left(4\pi^2\|\lambda\|_{\mathbf{m}_\rho}^2\right)^{-s} + \\
&\quad + \sum_{\lambda \in \Gamma_{\mathfrak{z}}^*(\rho)\backslash\{0\}} \sum_{p \in \mathbb{N}_0^n} \left(4\pi^2\|\lambda\|_{\mathbf{m}_\rho}^2 + 2\pi\|\lambda\|_{\mathbf{m}_\rho} \sum_j (2p_j+1)c_j^{\mathbf{m}_\rho}\right)^{-s} \\
&= \sum_{\lambda \in \Gamma_{\mathfrak{n}}^*(1)\backslash\{0\}} \rho^{-s/2n} \left(4\pi^2\|\lambda\|_{\mathbf{m}_1}^2\right)^{-s} + \\
&\quad + \sum_{\lambda \in \Gamma_{\mathfrak{z}}^*(1)\backslash\{0\}} \sum_{p \in \mathbb{N}_0^n} \left(\rho^{-1/\ell}4\pi^2\|\lambda\|_{\mathbf{m}_1}^2 + \rho^{1/2n}2\pi\|\lambda\|_{\mathbf{m}_1} \sum_j (2p_j+1)c_j^{\mathbf{m}_1}\right)^{-s}.
\end{aligned}$$

Obviously, this series is bounded from below by 0. We have a look at what happens when ρ goes to $+0$ or $+\infty$. In case $\rho \searrow 0$, $\rho^{-s/2n} \to \infty$ and hence $\zeta((\Gamma\backslash G, \mathbf{m}_\rho), s) \to \infty$.

If $\rho \to \infty$, then $\rho^{-s/2n} \to 0$. Also,

$$\rho^{-1/\ell}4\pi^2\|\lambda\|_{\mathbf{m}_1}^2 + \rho^{1/2n}2\pi\|\lambda\|_{\mathbf{m}_1} \sum_j (2p_j+1)c_j^{\mathbf{m}_1} \to \infty \quad \text{as} \quad \rho \to \infty\,,$$

and thus

$$\left(\rho^{-1/\ell}4\pi^2\|\lambda\|^2_{\mathbf{m}_1} + \rho^{1/2n}2\pi\|\lambda\|_{\mathbf{m}_1}\sum_j(2p_j+1)c_j^{\mathbf{m}_1}\right)^{-s} \to 0 \quad \text{as} \quad \rho \to \infty\,.$$

It follows that $\zeta((\Gamma\backslash G, \mathbf{m}_\rho), s) \to 0$ as $\rho \to \infty$. Summarising, we have seen that $[\mathbf{m}] \mapsto \zeta((\Gamma\backslash G, \mathbf{m}), s)$ is bounded from below by 0, but attains no global extremum on $\mathscr{S}\mathscr{M}^{HL}(\Gamma, G)$.

The last theorem shows that normalising the volume of both the base torus $T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}$ and the fibre torus $T_{\mathfrak{z},\mathbf{m}_\mathfrak{z}}$, which already excludes the path $\mathbf{m}_\rho$ from Proposition 2.12, does in general not enforce lower bounds for the height in dimensions $n \not\equiv 3 \pmod 4$. The question now is: what is a sufficient condition to guarantee the existence of lower bounds for the height? As the next theorem shows, a very natural one, namely that the sectional curvature is bounded from above, is such a condition.

Let $C > 0$ and recall from Definition 1.70:

$$\mathscr{S}\mathscr{M}^{HL}_C(\Gamma, G) := \{[\mathbf{m}] \in \mathscr{S}\mathscr{M}^{HL}(\Gamma, G) \mid K^{\mathbf{m}}_\sigma \leqslant C\}\,.$$

Note that by virtue of the path $\mathbf{m}_\rho$ defined in Proposition 2.12 we have $\mathscr{S}\mathscr{M}^{HL}_C(\Gamma, G) \neq \varnothing$ as soon as $\mathscr{S}\mathscr{M}^{HL}(\Gamma, G) \neq \varnothing$.

THEOREM 2.30. *Let G be connected and simply connected nonsingular 2-step nilpotent Lie group, $\Gamma \subset G$ a uniform subgroup and let $s \in (0,\infty)\backslash \operatorname{sing}(\dim G)$. Then*

(i) $[\mathbf{m}] \mapsto \zeta'((\Gamma\backslash G, \mathbf{m}), 0)$ *is bounded from below on* $\mathscr{S}\mathscr{M}^{HL}_C(\Gamma, G)$,

(ii) $[\mathbf{m}] \mapsto \zeta((\Gamma\backslash G, \mathbf{m}), s)$ *is bounded from below on* $\mathscr{S}\mathscr{M}^{HL}_C(\Gamma, G)$.

Furthermore, if the centre of G is one dimensional, then (i) and (ii) can be replaced by:

(iii) $[\mathbf{m}] \mapsto \zeta'((\Gamma\backslash G, \mathbf{m}), 0)$ *attains a global minimum on* $\mathscr{S}\mathscr{M}^{HL}_C(\Gamma, G)$,

(iv) $[\mathbf{m}] \mapsto \zeta((\Gamma\backslash G, \mathbf{m}), s)$ *attains a global minimum on* $\mathscr{S}\mathscr{M}^{HL}_C(\Gamma, G)$.

PROOF. Our proof will be analogous to that of Theorem 2.11. We start with a general remark. By definition $c^{\mathbf{m}}_n$ is the largest of the moduli of the eigenvalues of $j(Z)$, where $j_{\mathbf{m}} : \mathfrak{z} \to \mathfrak{so}(\mathfrak{n})$ is the structure map and $Z \in \mathfrak{z}$ with $\|Z\|_{\mathbf{m}} = 1$. By Proposition 1.12*(b)* we have $c^{\mathbf{m}}_n \leqslant 2\sqrt{C}$ on $\mathscr{S}\mathscr{M}^{HL}_C(\Gamma, G)$.

Choose $N := \lfloor \dim G/2 \rfloor$ in *(i)* and $N \in \mathbb{N}_0 \cup \{-1\}$ such that $N > \dim G/2 - s - 1$ in *(ii)*. By Theorem 2.5 and Corollary 2.6 the functions $[\mathbf{m}] \mapsto \Gamma(s)\zeta((\Gamma\backslash G, \mathbf{m}), s)$ and $[\mathbf{m}] \mapsto \zeta'((\Gamma\backslash G, \mathbf{m}), 0)$ are of the form

$$
\begin{aligned}
(2.50)\quad & C(\tau) + \int_1^\infty \sum_{\lambda\in\Gamma_\mathfrak{n}^*\setminus\{0\}} e^{-4\pi^2\|\lambda\|^2_{\mathbf{m}_\mathfrak{n}} t} t^{\tau-1}\mathrm{d}t + \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n} \int_1^\infty \sum_{X\in\Gamma_\mathfrak{n}\setminus\{0\}} e^{-\frac{\|X\|^2_{\mathbf{m}_\mathfrak{n}}}{4}t} t^{n-\tau-1}\mathrm{d}t \\
& + \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n} \int_1^\infty \sum_{\lambda\in\Gamma_\mathfrak{z}^*\setminus\{0\}} s^{\mathbf{m}}_{V,t}(\lambda) t^{\tau-n-1}\mathrm{d}t + \frac{1}{(4\pi)^{\dim G/2}} \int_0^1 \sum_{X\in\Gamma_\mathfrak{z}\setminus\{0\}} \sigma^{\mathbf{m}}_X(t) t^{\tau-\dim G/2-1}\mathrm{d}t \\
& + \frac{1}{(4\pi)^{\dim G/2}} \left(\int_0^1 \left(\sigma^{\mathbf{m}}_0(t) - T^N_{+0}[\sigma^{\mathbf{m}}_0](t)\right) t^{\tau-\dim G/2-1}\mathrm{d}t + \sum_{\substack{j=0\\ j\neq \dim G/2}}^{N} \frac{a^{\mathbf{m}}_j}{\tau - \dim G/2 + j} + \right. \\
& \qquad\qquad \left. + \begin{cases} s^{-1}\cdot a^{\mathbf{m}}_{\dim G/2} & \text{if } \dim G \text{ is even and } \tau = s\,, \\ \gamma\cdot a^{\mathbf{m}}_{\dim G/2} & \text{if } \dim G \text{ is even and } \tau = 0\,, \\ 0 & \text{if } \dim G \text{ is odd} \end{cases} \right),
\end{aligned}
$$

where τ equals 0 or s respectively, $C(\tau)\in\mathbb{R}$ is a constant and $T^N_{+0}[\sigma^{\mathbf{m}}_0](t) = \sum_{j=0}^N a^{\mathbf{m}}_j t^j$.

The first line of (2.50) is clearly bounded from below. The first term in the second line of (2.50) is positive by definition of $s^{\mathbf{m}}_{V,t}$, see (1.42). The second term in that line is positive by Lemma 1.76. We address the sum in the third line, including the therm in the fourth line. Each $a^{\mathbf{m}}_j$ is a polynomial in $c^{\mathbf{m}}_1,\dots,c^{\mathbf{m}}_n$. We have seen at the beginning of the proof that $c^{\mathbf{m}}_n$, and hence all $c^{\mathbf{m}}_j$, are bounded from above. Hence, the whole sum is bounded. We now have a look at the integral in the third line of (2.50). By Taylor's Theorem with Lagrangian remainder we have

$$
\left|\sigma^{\mathbf{m}}_0(t) - T^N_{+0}[\sigma^{\mathbf{m}}_0](t)\right| t^{\tau-\dim G/2-1} \leqslant \frac{1}{(N+1)!}\max_{u\in[0,1]}\left|\frac{\mathrm{d}^{N+1}}{\mathrm{d}u^{N+1}}\sigma^{\mathbf{m}}_0(u)\right| \cdot t^{N+\tau-\dim G/2}.
$$

The right hand side is integrable on $t\in[0,1]$ for every $\mathbf{m}\in\mathscr{SM}^{HL}_C(\Gamma,G)$ by choice of N. By boundedness of $c^{\mathbf{m}}_n$ on $\mathscr{SM}^{HL}_C(\Gamma,G)$ and Lemma 1.88 the right hand side is bounded as a function of $[\mathbf{m}]\in\mathscr{SM}^{HL}_C(\Gamma,G)$. Hence, the integral in the third line of (2.50) is bounded, in particular from below. This finishes the proof of *(i)* and *(ii)*.

We will now prove *(iii)* and *(iv)*. Since the centre of G is hypothesised to be one dimensional, we can w.l.o.g. assume that $G = H_n$ and $\Gamma = \Gamma^r$ for some $r\in\mathcal{D}_n$ (see Theorem 1.44). Also, we switch to the moduli space $\mathcal{SM}^r_{n,C}\simeq\mathscr{SM}^{HL}_C(\Gamma^r,H_n)$. The argument that lies at the core of our proof is the same as that of Theorem 2.11*(iii)* and *(iv)*. On every compact set $M\subset\mathcal{SM}^r_{n,C}$, (2.50) attains a minimum. To prove that (2.50) attains a global minimun it thus suffices to show that (2.50) goes to $+\infty$ as $[\mathbf{m}] = [(h,g)]\to\partial_\infty\mathcal{SM}^r_{n,C}$, the boundary at infinity of $\mathcal{SM}^r_{n,C}$. It is defined by the degeneracies $m_r(h)=0$, $\det(h)=\infty$, $d_n(h)=\infty$ and $g\in\{0,\infty\}$, see Remark 1.66. We will first show that $\partial_\infty\mathcal{SM}^r_{n,C}$

can be described by the following two alternative degeneracies: $m(h^{-1}[\delta_r^{-1}]) = 0; g < b < \infty$ and $m_r(h) = 0$.

We will go through the defining degeneracies of $\partial_\infty \mathcal{SM}^r_{n,C}$ one by one and show that they can either not occur or imply one of the two alternative ones described above. Note that every $[(h,g)] \in \mathcal{SM}^r_{n,C}$ satisfies two constraints: $g \cdot \det h \cdot |\Gamma^r|^2 = 1$ and $c_n^{(h,g)} = g^{1/2} d_n(h) \leqslant C$.

(a) Suppose $\det(h) \to \infty$. Then clearly $\det(h^{-1}) \to 0$. By Remark 1.55 we then have $m(h^{-1}) \to 0$, which in turn implies $m(h^{-1}[\delta_r^{-1}]) \to 0$. This is covered by the first case.
(b) Suppose now that $d_n(h) \to \infty$. By the constraint $g^{1/2} d_n(h) \leqslant C$, we then have $g \to 0$, which implies $\det(h) = |\Gamma^r|^{-2} \cdot g^{-1} \to \infty$. As explained in the previous case, this implies $m(h^{-1}[\delta_r^{-1}]) \to 0$.
(c) In case $g \to 0$ we have again $\det(h) = |\Gamma^r|^{-2} \cdot g^{-1} \to \infty$, which implies $m(h^{-1}[\delta_r^{-1}]) \to 0$ as in *(a)*.
(d) At last, if $g \to \infty$ then $\det(h) = |\Gamma^r|^{-2} \cdot g^{-1} \to 0$ and hence
$$\det(h^{-1}) = d_1(h)^2 \cdots d_n(h)^2 \to \infty\,.$$
This implies $d_n(h) \to \infty$. But then $g^{1/2} \cdot d_n(h) \to \infty$, which cannot happen since $g^{1/2} \cdot d_n(h) \leqslant C$.

Now we look at the growth of the first line of (2.50) in case 1.) $m(h^{-1}[\delta_r^{-1}]) \to 0$ or 2.) $m_r(h) \to 0$ and g is bounded from above.

In case $m\left(h^{-1}\left[\delta_r^{-1}\right]\right) \to 0$ we rewrite the first integral in the first line of (2.50) as

$$\int_1^\infty \sum_{X \in \mathbb{Z}^{2n} \setminus \{0\}} e^{-4\pi^2 h^{-1}[\delta_r^{-1}][X]} t^{\tau-1} \mathrm{d}t\,. \tag{2.51}$$

Here, we used $\Gamma^r_{\mathrm{n}} = \delta_r \cdot \mathbb{Z}^{2n}$ (see Proposition 1.49) and the fact that $h = \mathbf{m}_{\mathrm{n}}$. It now follows from Lemma 2.9 that (2.51) goes to $+\infty$ as $m\left(h^{-1}\left[\delta_r^{-1}\right]\right) \to 0$.

In case g is bounded from above and $m_r(h) \to 0$ we rewrite the last term in the first line of (2.50) as

$$\frac{\det h^{1/2} \cdot |\Gamma^r|}{(4\pi)^n} \int_1^\infty \sum_{X \in \mathbb{Z}^{2n} \setminus \{0\}} e^{-\frac{h[\delta_r][X]}{4} t} t^{n-\tau-1} \mathrm{d}t\,, \tag{2.52}$$

where we used again $\Gamma^r_{\mathrm{n}} = \delta_r \cdot \mathbb{Z}^{2n}$ and $h = \mathbf{m}_n$. By the constraint $g \cdot \det h \cdot |\Gamma^r|^2 = 1$ and the assumption that g is bounded from above, $\det h$ is bounded from below. Since $m_r(h) = m(h[\delta_r])$ the right hand side goes to $+\infty$ when $m_r(h) \to 0$ by Lemma 2.9. $\square$

In the following example we define uniform groups $\Gamma^{n,\ell} \subset H_n$ for every $n, \ell \in \mathbb{N}$. In **[Fd03]**, K. Furutani und S. de Gosson investigated the height of the Heisenberg manifolds

$(\Gamma^{1,\ell}\backslash H, \mathbf{m}_{std})$ and $(\Gamma^{1,\ell}\backslash H, \mathbf{m}_{std})$, where $\mathbf{m}_{std}$ is the standard metric of $\mathfrak{h}_n \simeq \mathbb{R}^{2n+1}$. They proved that the height of these manifolds goes to ∞ as $\ell \to \infty$. We generalise this result to all dimensions in Proposition 2.32.

EXAMPLE 2.31 (cf. **[Fd03]** for the cases $n = 1, 2$). Let $n, \ell \in \mathbb{N}$ and define the uniform subgroup $\Gamma^{n,\ell} \subset H_n$ by

$$\Gamma^{n,\ell} := \left\{ \gamma\left(x, y, \frac{k}{2\ell}\right) \,\Big|\, x, y \in \mathbb{Z}^n, k \in \mathbb{Z} \right\} = \left\{ \begin{pmatrix} 1 & {}^{t}x & k/2\ell \\ 0 & \mathrm{Id}_n & y \\ 0 & 0 & 1 \end{pmatrix} \,\Big|\, x, y \in \mathbb{Z}^n, k \in \mathbb{Z} \right\}.$$

Let $\mathbf{m} = \mathbf{m}_{std}$ be the standard metric of $\mathfrak{h}_n$ in which the standard basis $\mathfrak{B}_n$ of $\mathbb{R}^{2n+1}$ (see Example 1.2 and also (1.21)) is an orthonormal basis. Then $\mathbf{m}_{\mathfrak{z}}$ and $\mathbf{m}_{\mathfrak{n}}$ are the standard inner products on $\mathbb{R}$ and $\mathbb{R}^{2n}$ respectively. By (1.7) on page 11 we have

$$\log \Gamma^{n,\ell} = \{\log \gamma(x, y, k/2\ell) \,|\, x, y \in \mathbb{Z}^n, k \in \mathbb{Z}\} = \{X(x, y, k/2\ell - 1/2\langle x, y\rangle_{std}) \,|\, x, y \in \mathbb{Z}^n, k \in \mathbb{Z}\}.$$

It follows that

- $\Gamma^{n,\ell}_{\mathfrak{z}} = \log \Gamma^{n,\ell} \cap \mathfrak{z} = \frac{1}{2\ell}\mathbb{Z} \cdot Z$ and $\mathrm{Vol}\, T_{\mathfrak{z},(\mathbf{m}_{std})_{\mathfrak{z}}} = \mathrm{Vol}(\Gamma^{n,\ell}_{\mathfrak{z}}\backslash\mathbb{R}, 1) = 1/2\ell$,
- $\Gamma^{n,\ell}_{\mathfrak{n}} = \pi_{\mathfrak{n}}\left(\log \Gamma^{n,\ell}\right) = \mathbb{Z}^{2n}$ and $\mathrm{Vol}\, T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}} = \mathrm{Vol}(\Gamma^{n,\ell}_{\mathfrak{n}}\backslash\mathbb{R}^{2n}, \mathbf{m}_{\mathfrak{n}}) = 1$,
- $\mathrm{Vol}(\Gamma^{n,\ell}\backslash H_n, \mathbf{m}_{std}) = \mathrm{Vol}\, T_{\mathfrak{z},(\mathbf{m}_{std})_{\mathfrak{z}}} \cdot \mathrm{Vol}\, T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}} = 1/2\ell$,
- $c^{\mathbf{m}}_k = 1$ for all $1 \leqslant k \leqslant n$ since $\langle j(Z)X_i, Y_j\rangle_{\mathbf{m}_{\mathfrak{n}}} = \langle Z, [X_i, Y_j]\rangle_{\mathbf{m}_{\mathfrak{z}}} = \delta_{i,j}$ and $[X_i, X_j] = [Y_i, Y_j] = 0$, where $\delta_{i,j}$ is the Kronecker-delta symbol.

PROPOSITION 2.32 (cf. **[Fd03**, Corollaries 3.9, 5.5] for the cases $n = 1, 2$). *Let $n \in \mathbb{N}$. Then*

$$\zeta'((\Gamma^{n,\ell}\backslash H_n, \mathbf{m}_{std}), 0) \to \infty \;\; as \;\; \ell \to \infty.$$

PROOF. We work with the formula for $\zeta'((\Gamma^{n,\ell}\backslash H_n, \mathbf{m}_{std}), 0)$ given in Corollary 2.6. Since $(\mathbf{m}_{std})_{\mathfrak{n}}$ is simply the standard inner product on $\mathfrak{n} = \mathbb{R}^{2n}$ and $\Gamma^{n,\ell}_{\mathfrak{n}}$ is independent of ℓ, $\zeta'_B((\Gamma^{n,\ell}\backslash H_n, \mathbf{m}_{std}), 0)$ is independent of ℓ and thus constant as a sequence in ℓ. We have a look at formula (2.15) for $\zeta'_F((\Gamma^{n,\ell}\backslash H_n, \mathbf{m}_{std}), 0)$:

(2.53)
$$\frac{\mathrm{Vol}\, T_{\mathfrak{n},(\mathbf{m}_{std})_{\mathfrak{n}}}}{(4\pi)^n} \int_1^\infty \sum_{\lambda \in \Gamma^*_{\mathfrak{z}}\setminus\{0\}} s^{\mathbf{m}_{std}}_{V,t}(\lambda) t^{-n-1} \mathrm{d}t + \frac{\mathrm{Vol}\left(\Gamma^{n,\ell}\backslash H_n, \mathbf{m}_{std}\right)}{(4\pi)^{(2n+1)/2}} \int_0^1 \sum_{X \in \Gamma_{\mathfrak{z}}\setminus\{0\}} \sigma^{\mathbf{m}_{std}}_X(t) t^{-(2n+1)/2-1} \mathrm{d}t$$
$$+ \frac{\mathrm{Vol}(\Gamma^{(n,\ell)}\backslash H_n, \mathbf{m}_{std})}{(4\pi)^{(2n+1)/2}} \left(\int_0^1 \left(\sigma^{\mathbf{m}_{std}}_0(t) - \sum_{j=0}^N a^{\mathbf{m}_{std}}_j t^j \right) t^{-(2n+1)/2-1} \mathrm{d}t + \sum_{j=0}^N \frac{a^{\mathbf{m}_{std}}_j}{j - \frac{\dim G}{2}} \right).$$

The second line of (2.53) goes to 0 as $\ell \to \infty$; indeed, $\mathrm{Vol}(\Gamma^{(n,\ell)}\backslash H_n, \mathbf{m}) = 1/2\ell \to 0$ as $\ell \to \infty$ and the expression in parentheses is constant in ℓ, since it only depends on $c^{\mathbf{m}}_j$,

$1 \leqslant j \leqslant n$, and these are 1 (see (1.43) for the definition of $\sigma_0^{\mathbf{m}}$). The first line of (2.53) is positive by definition of $s_{V,t}^{\mathbf{m}}$ (see (1.42)) and Corollary 1.84. We will show that the second term in this line goes to $+\infty$ as $\ell \to \infty$.

Since $\Gamma_{\mathfrak{z}}^{n,\ell} = \frac{1}{2\ell}\mathbb{Z}$ we have (see (1.43) for the definition of $\sigma_X^{\mathbf{m}}$), writing $\mathbf{m}$ for $\mathbf{m}_{std}$:

$$\frac{\operatorname{Vol}\left(\Gamma^{n,\ell}\backslash H_n, \mathbf{m}\right)}{(4\pi)^{(2n+1)/2}} \int_0^1 \sum_{X \in \Gamma_{\mathfrak{z}} \setminus \{0\}} \sigma_X^{\mathbf{m}}(t) \cdot t^{-(2n+1)/2-1} \mathrm{d}t = \frac{(2\ell)^{-1}}{(4\pi)^{(2n+1)/2}} \int_0^1 \sum_{c \in \mathbb{Z} \setminus \{0\}} \sigma_{c/2\ell}^{\mathbf{m}}(t) \cdot t^{-(2n+1)/2-1} \mathrm{d}t$$

$$= \frac{(2\ell)^{-1}}{(4\pi)^{(2n+1)/2}} \int_0^1 \sum_{c \in \mathbb{Z} \setminus \{0\}} \int_{\mathbb{R}} e^{-\frac{i}{\sqrt{t}} \frac{c}{2\ell} \xi} e^{-\xi^2} \left(\frac{\sqrt{t}\xi}{\sinh(\sqrt{t}\xi)} \right)^n \mathrm{d}\xi \cdot t^{-(2n+1)/2-1} \mathrm{d}t$$

$$= \frac{(2\ell)^{2n}}{(4\pi)^{(2n+1)/2}} \int_0^{4\ell^2} \sum_{c \in \mathbb{Z} \setminus \{0\}} \int_{\mathbb{R}} e^{-\frac{i}{\sqrt{t}} c \xi} e^{-\xi^2} \left(\frac{\sqrt{t}/2\ell\xi}{\sinh(\sqrt{t}/2\ell\xi)} \right)^n \mathrm{d}\xi \cdot t^{-(2n+1)/2-1} \mathrm{d}t$$

$$\geqslant \frac{(2\ell)^{2n}}{(4\pi)^{(2n+1)/2}} \int_0^1 \sum_{c \in \mathbb{Z} \setminus \{0\}} \int_{\mathbb{R}} e^{-\frac{i}{\sqrt{t}} c \xi} e^{-\xi^2} \left(\frac{\sqrt{t}/2\ell\xi}{\sinh(\sqrt{t}/2\ell\xi)} \right)^n \mathrm{d}\xi \cdot t^{-(2n+1)/2-1} \mathrm{d}t\,.$$

The same arguments that we used in Theorem 1.78, Lemma 2.15 and Lemma 2.24 show that the integral converges to a finite number; i.e., one first uses Theorem 1.36 to show that $\lim_{\ell \to \infty}$ commutes with the inner integral. Then one shows that for every $N \in \mathbb{N}$ there exists $C > 0$ such that $|\int_{\mathbb{R}} \ldots \mathrm{d}\xi| \leqslant C \cdot t^N$ for all $t \in [0,1]$. This implies normal convergence of the series and boundedness of the integrand of the outer integral. It then follows, again by Theorem 1.36, and by continuity of normally convergent series, that $\lim_{\ell \to \infty}$ commutes also with the outer integral and the series. Hence, $\zeta'((\Gamma^{n,\ell}\backslash H_n, \mathbf{m}_{std}), 0) \to \infty$ as $\ell \to \infty$. $\square$

3. Extremal Metrics

This last section discusses extremal metrics for the height and the ζ-function. In Section 3.1 we look at flat tori and introduce the *Lattice Packing Problem*, which consists of finding lattices which maximise a certain notion of density. We then present work of P. Sarnak et al. that shows that in dimensions $n = 2, 4, 8, 24$ the height and the ζ-function of flat tori of volume one are always minimised at tori $L\backslash\mathbb{R}^n$, where L is *special* in that it solves the lattice packing problem.

In Section 3.2 we use these results to show that in dimensions $\ell + 2n = 1 + 2n = 3, 5, 9, 25$ the height and the ζ-function of compact Heisenberg-type Heisenberg manifolds $(\Gamma^{\mathbb{1}}\backslash H^n, \mathbf{m})$ are minimised at those metrics for which the base torus $T_{\mathfrak{n},\mathbf{m}_{\mathfrak{n}}}$ comes from a special lattice, see Theorem 2.49.

Theorem 2.50 discusses a neccessary criterion for a metric to be extremal: If the metric $[(h_0, g_0)] \in \mathcal{SM}_n^r$ is an extremal (a local or global (non-)strict minimum or maximum) metric for ζ' then h_0 must be an extremal metric of the same type for the height of flat tori on a certain submanifold of the moduli space of flat tori.

This enables us to identify the global minimiser(s) of the height of all 3-dimensional nilmanifolds. The corresponding result is stated in Theorem 2.52. We finish with a discussion of the 5 dimensional case.

3.1. Epstein's ζ-Function and the Height of Flat Tori.

Recall (2.1) from Definition 2.1:

$$D(\alpha) = \{s \in \mathbb{C} \mid \Re s > \alpha\} .$$

DEFINITION 2.33. Let $n \in \mathbb{N}$. Define Epstein's ζ-function $\zeta_{Ep} : \mathcal{P}_n \times D(n/2) \to \mathbb{C}$ by

$$(Y, s) \mapsto \sum_{m \in \mathbb{Z}^n \setminus \{0\}} (Y[m])^{-s} . \tag{2.54}$$

The Dirichlet series on the right converges absolutely for any $Y \in \mathcal{P}_n$, $s \in D(n/2)$ (see **[Ter88**, Chapter 4.4, Corollary 2]).

REMARK 2.34. Let $\lambda_1 : \mathcal{P}_n \to \mathbb{R}$ be the function such that $\lambda_1(Y)$ is the smallest eigenvalue of $Y \in \mathcal{P}_n$. Note that $\lambda_1(Y)\mathrm{Id}[m] \leqslant Y[m]$ for all $m \in \mathbb{Z}^n$ and that λ_1 is continuous (see Notation and Remarks 1.61*(i)*). We thus have

$$\sum_{m \in \mathbb{Z}^n \setminus \{0\}} \left|(Y[m])^{-s}\right| = \sum_{m \in \mathbb{Z}^n \setminus \{0\}} (Y[m])^{-\Re s} \leqslant \sum_{m \in \mathbb{Z}^n \setminus \{0\}} \lambda_1(Y)^{-\Re s} (\mathrm{Id}[m])^{-\Re s} .$$

It follows that the series on the right-hand side of (2.54) converges normally on every set $K \times \overline{D(\delta)}$, where $K \subset \mathcal{P}_n$ is such that λ_1 is bounded away from 0 and $\delta > n/2$.

REMARK 2.35. The group $\mathrm{GL}(n; \mathbb{Z})$ acts on $\mathcal{P}_n$ via

$$\begin{aligned} \mathcal{P}_n \times \mathrm{GL}(n; \mathbb{Z}) &\to \mathcal{P}_n \\ (Y, G) &\mapsto Y[G] = {}^{\mathrm{t}}G \cdot Y \cdot G . \end{aligned}$$

Since $G \cdot \mathbb{Z}^n = \mathbb{Z}^n$ for all $G \in \mathrm{GL}(n; \mathbb{Z})$, the Dirichlet series in (2.54) is invariant under the action of $\mathrm{GL}(n; \mathbb{Z})$ on $\mathcal{P}_n$. Thus, $Y \mapsto \zeta_{Ep}(Y, s)$ descends to a function on $\mathcal{P}_n / \mathrm{GL}(n; \mathbb{Z})$.

THEOREM 2.36 (**[Eps03**, §1], **[Ter85**, Chapter 1.4, Theorem 1]). *Let $n \in \mathbb{N}$ and fix $Y \in \mathcal{P}_n$. As a function of $s \in D(n/2)$ Epstein's ζ-function has a meromorphic continuation to all of $\mathbb{C}$. This continuation is given by*

$$\pi^{-s}\Gamma(s)\zeta_{Ep}(Y,s) = \frac{\det Y^{-1/2}}{s - {}^{n}/_{2}} - \frac{1}{s} + \tag{2.55}$$
$$+ \int_1^\infty \sum_{m\in\mathbb{Z}^n\setminus\{0\}} e^{-\pi Y[m]t} t^{s-1} \mathrm{d}t + \frac{1}{\sqrt{\det Y}} \int_1^\infty \sum_{m\in\mathbb{Z}^n\setminus\{0\}} e^{-\pi Y^{-1}[m]t} t^{n/2-s-1} \mathrm{d}t\,.$$

From (2.55) *it is immediate that* $\zeta_{Ep}(Y,\cdot)$ *has but one pole at* $s = {}^{n}/_{2}$ *with residue*

$$\det Y^{-1/2}\pi^{n/2}\Gamma({}^{n}/_{2})^{-1}\,.$$

Furthermore, (2.55) *shows that* ζ_{Ep} *satisfies the functional equation*

$$\pi^{-s}\Gamma(s)\zeta_{Ep}(Y,s) = \frac{\pi^{s-n/2}\Gamma({}^{n}/_{2}-s)}{\sqrt{\det Y}}\zeta_{Ep}\left(Y^{-1}, \tfrac{n}{2}-s\right)\,. \tag{2.56}$$

REMARK 2.37. Note that by (2.54) we have

$$\zeta_{Ep}(a\cdot Y, s) = a^{-s}\cdot\zeta_{Ep}(Y,s)$$

for all $a \in (0,\infty)$, $Y \in \mathcal{P}_n$ and $s \in D({}^{n}/_{2})$ and by meromorphic continuation for all $s \in \mathbb{C}\setminus\{{}^{n}/_{2}\}$. Differentiating this identity in $s = 0$ yields

$$\zeta'_{Ep}(a\cdot Y,0) = -\ln a\cdot\zeta_{Ep}(Y,0) + \zeta'_{Ep}(Y,0)\,.$$

Formula (2.55) implies that $\zeta_{Ep}(Y,0) = -1$. Hence,

$$\zeta'_{Ep}(a\cdot Y,0) = \ln a + \zeta'_{Ep}(Y,0)\,.$$

OBSERVATION 2.38. Paul Epstein introduced his ζ-function **[Eps03]** with the intention to generalise Riemann's ζ-function. Our interest in ζ_{Ep} lies in it being the spectral ζ-function of flat tori. Let $L \subset \mathbb{R}^n$ be a lattice of full rank, $(v_1,\ldots,v_n)$ a $\mathbb{Z}$-basis of L and Y the corresponding Gram matrix. Then Y^{-1} is a Gram matrix for the dual lattice $L^* = \{\lambda \in \mathbb{R}^n \mid \langle\lambda,v\rangle_{std} \in \mathbb{Z} \text{ for all } v \in L\}$. The eigenvalues of the Laplace-Beltrami operator on the flat torus $T := (L\backslash\mathbb{R}^n, \langle\cdot,\cdot\rangle_{std}) \simeq (\mathbb{Z}^n\backslash\mathbb{R}^n, Y)$ are

$$4\pi^2\|\lambda\|^2, \quad \lambda \in L^*\,, \tag{2.57}$$

which are the same as

$$4\pi^2 Y^{-1}[m], \quad m \in \mathbb{Z}^n\,.$$

We thus see that the spectral ζ-function of T is given by

$$\zeta(T,s) = \sum_{\lambda\in L^*\setminus\{0\}} \left(4\pi^2\|\lambda\|^2\right)^{-s} = \zeta_{Ep}\left(4\pi^2 Y^{-1}, s\right) = (4\pi^2)^{-s}\zeta_{Ep}\left(Y^{-1},s\right)\,. \tag{2.58}$$

In particular, the height $\zeta'(T,0) = \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} = \zeta(T,s)$ of the flat torus T is

$$\zeta'(T,0) = 2\ln 2\pi + \zeta'_{Ep}\left(Y^{-1},0\right)\,. \tag{2.59}$$

By virtue of equations (2.58) and (2.59), we view the spectral ζ-function and the height of flat tori as functions on $\mathcal{P}_n / \operatorname{GL}(n;\mathbb{Z})$.

We now look at an arbitrary compact nonsingular Heisenberg-like nilmanifold $(\Gamma\backslash G, \mathbf{m})$. Let $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{n}$ be the Lie algebra of G, where $\mathfrak{n} = \mathfrak{z}^\perp$ is the $\mathbf{m}$-orthogonal complement of the centre $\mathfrak{z}$. The fibres of the Riemannian submersion $\pi : (\Gamma\backslash G, \mathbf{m}) \to T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}$ (see Proposition 1.10) onto the base torus $T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}} = (\Gamma_\mathfrak{n}\backslash\mathfrak{n}, \mathbf{m}_\mathfrak{n})$ are totally geodesic. Hence the spectrum of $T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}$ is part of the spectrum of $(\Gamma\backslash G, \mathbf{m})$, which can also be seen directly in Corollary 1.14. This fact is reflected in the spectral ζ-function.

Recall that we split the spectral ζ-function of $(\Gamma\backslash G, \mathbf{m})$ into two parts ζ_B and ζ_F in Theorem 2.5. Choose an orthonormal basis of $(\mathfrak{n}, \mathbf{m}_\mathfrak{n})$ and identify $\mathfrak{n}$ with $\mathbb{R}^{2n}$ by a linear isometry. Let $M \in \operatorname{GL}(2n;\mathbb{R})$ be a generator matrix for $\Gamma_\mathfrak{n} = \pi_\mathfrak{n} \log \Gamma$ and $Y = \operatorname{Id}[M]$ the corresponding Gram matrix, i.e., $\Gamma_\mathfrak{n} = M \cdot \mathbb{Z}^{2n}$ and $T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}$ is isometric to $(\mathbb{Z}^{2n}\backslash\mathbb{R}^{2n}, Y)$. Then we have by Theorem 2.36, (2.58) and the definition of ζ_B in (2.8):

$$
\begin{aligned}
(2.60)\quad & \Gamma(s)\zeta_B\left((\Gamma\backslash G, \mathbf{m}), s\right) = \\
& -\frac{1}{s} + \int_1^\infty \sum_{\lambda \in \Gamma_\mathfrak{n}^* \setminus \{0\}} e^{-4\pi^2 \|\lambda\|^2 t} t^{s-1} \mathrm{d}t + \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n} \int_1^\infty \sum_{X \in \Gamma_\mathfrak{n} \setminus \{0\}} e^{-\frac{\|X\|^2}{4} t} t^{n-s-1} \mathrm{d}t = \\
& -\frac{1}{s} + \int_1^\infty \sum_{m \in \mathbb{Z}^{2n} \setminus \{0\}} e^{-4\pi^2 Y^{-1}[m] t} t^{s-1} \mathrm{d}t + \det\left(4\pi Y^{-1}\right)^{-1/2} \int_1^\infty \sum_{m \in \mathbb{Z}^{2n} \setminus \{0\}} e^{-\frac{Y}{4}[m] t} t^{n-s-1} \mathrm{d}t \\
& = \pi^{-s}\Gamma(s)\zeta_{Ep}(4\pi Y^{-1}, s) - \frac{\det\left(4\pi Y^{-1}\right)^{-1/2}}{s-n} \\
& = (4\pi^2)^{-s}\Gamma(s)\zeta_{Ep}(Y^{-1}, s) - \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n} \cdot \frac{1}{s-n} \\
& = \Gamma(s)\zeta(T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}, s) - \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n} \cdot \frac{1}{s-n}\,.
\end{aligned}
$$

The last term cancels the only pole of $\zeta_{Ep}(4\pi Y^{-1}, s)$ and $\zeta(T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}, s)$ respectively. Differentiating (2.60) in $s = 0$ yields by (2.59)

$$
\begin{aligned}
(2.61)\quad \zeta_B'\left((\Gamma\backslash G, \mathbf{m}), 0\right) &= 2\ln 2\pi + \zeta_{Ep}'(Y^{-1}, 0) + \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n \cdot n} \\
&= \zeta'(T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}, 0) + \frac{\operatorname{Vol} T_{\mathfrak{n},\mathbf{m}_\mathfrak{n}}}{(4\pi)^n \cdot n}\,.
\end{aligned}
$$

DEFINITION AND REMARKS 2.39.

(i) We introduce certain special lattices in dimensions $n = 2, 4, 8, 24$. The first is the two dimensional *hexagonal lattice* A_2. A generator matrix for A_2 (see [**CS99**, Ch. 4.6.2]) is

$$M_{A_2} := \begin{pmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix} .$$

In this representation the volume of A_2 is $\det M_{A_2} = \sqrt{3}/2$. After normalising the volume we obtain the following Gram matrix of determinant 1 for A_2:

$$Y_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} . \tag{2.62}$$

The next is the four dimensional *checkerboard lattice* $D_4 := \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_1 + x_2 + x_3 + x_4 \equiv 0 \pmod 2\}$ (see [**CS99**, Ch. 4.7.2] and Remark 2.41(ii)) for which a generator matrix is given by

$$M_{D_4} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} .$$

Here the volume is 2 and after normalization we obtain the corresponding Gram matrix

$$Y_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 1 & 0 & -1 & 2 \end{pmatrix} . \tag{2.63}$$

A special eight dimensional lattice is $E8 := \{(x_1, \ldots, x_8) \mid \text{all } x_i \in \mathbb{Z} \text{ or } x_i \in \mathbb{Z} + \frac{1}{2}, \sum_i x_i \equiv 0 \pmod 2\}$ (see [**CS99**, Ch. 4.8.1]), which is sometimes called the *8-dimensional diamond lattice*. A generator matrix is given by

$$M_{E_8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix} . \tag{2.64}$$

Here, $\operatorname{Vol} E_8 = 1$. For the definition of the last special lattice we consider, we first need to define the *Golay code* $\mathscr{G}_{24}$ (see [**CS99**, p. 84]). Its code words are those $\epsilon = (\epsilon_1, \ldots, \epsilon_{24}) \in \mathbb{Z}_2^{24}$ which are in the linear span of the rows of the matrix

$$\begin{bmatrix}
1 &&&&&&&&&&& & 1&0&1&0&0&0&1&1&1&0&1&1 \\
& 1 &&&&&&&&&& & 1&1&0&1&0&0&0&1&1&1&0&1 \\
&& 1 &&&&&&&&& & 0&1&1&0&1&0&0&0&1&1&1&1 \\
&&& 1 &&&&&&&& & 1&0&1&1&0&1&0&0&0&1&1&1 \\
&&&& 1 &&&&&&& & 1&1&0&1&1&0&1&0&0&0&1&1 \\
&&&&& 1 &&&&&& & 1&1&1&0&1&1&0&1&0&0&0&1 \\
&&&&&& 1 &&&&& & 0&1&1&1&0&1&1&0&1&0&0&1 \\
&&&&&&& 1 &&&& & 0&0&1&1&1&0&1&1&0&1&0&1 \\
&&&&&&&& 1 &&& & 0&0&0&1&1&1&0&1&1&0&1&1 \\
&&&&&&&&& 1 && & 1&0&0&0&1&1&1&0&1&1&0&1 \\
&&&&&&&&&& 1 & & 0&1&0&0&0&1&1&1&0&1&1&1 \\
&&&&&&&&&&& 1 & 1&1&1&1&1&1&1&1&1&1&1&0
\end{bmatrix}.$$

Here, empty spaces stand for 0. Now, the Leech lattice Λ_{24} is defined as the lattice generated by all vectors of the form

$$\frac{1}{\sqrt{8}}((-1)^{\epsilon_1}x_1, \ldots, (-1)^{\epsilon_{24}}x_{24}),$$

with exactly one $x_j = -3$, $x_k = 1$ for all $1 \leqslant k \leqslant 24$ with $k \neq j$ and $\epsilon \in \mathscr{G}_{24}$ (see [**CS99**, p. 131]). We remark that the Leech lattice has a surprisingly large number of equivalent definitions, see [**CS99**]. In this representation, the volume of Λ_{24} is one, see [**CS99**, p. 133].

(ii) For any lattice $L \subset \mathbb{R}^n$, there is an associated *lattice packing*. This is the arrangement of n-dimensional pairwise congruent, nonoverlapping but touching balls with centres the lattice points. Since the balls are not allowed to intersect but must touch, their radius r (which is also called the *packing radius*) must be equal to one half of the length of a shortest nonzero vector in L; that is, $r = \frac{1}{2}m(Y)^{1/2}$, where Y is a Gram matrix for L. An important invariant of a lattice packing is its *density* Δ, which is defined as the proportion of $\mathbb{R}^n$ that is occupied by the balls. The density is given by the formula

$$\Delta = \frac{\operatorname{Vol} B\left(\frac{1}{2}m(Y)^{1/2}\right)}{\operatorname{Vol} L} = \frac{V_n 2^{-n} m(Y)^{n/2}}{\det Y^{1/2}}, \tag{2.65}$$

where $B\left(\frac{1}{2}m(Y)^{1/2}\right)$ stands for the n-dimensional ball with centre 0, radius $\frac{1}{2}m(Y)^{1/2}$ and volume $V_n := \operatorname{Vol} B(1)$. To see (2.65), let $(v_1, \ldots, v_n)$ be a basis of L and Y the corresponding Gram matrix. The parallelotope P consisting of all points $\sum_{j=0}^{n} \theta_j v_j$ with $0 \leqslant \theta_j < 1$ for all j is a fundamental domain for L and there is exactly one point of L that is contained in P, namely 0. This implies that

$$\mathbb{R}^n = \bigcup_{v \in L} (v + P)\,.$$

On the other hand, the set

$$\bigcup_{v \in L} \left(v + B\left(\tfrac{1}{2} m(Y)^{1/2}\right)\right)$$

is precisely the lattice packing associated with L, so that

$$\Delta = \frac{\operatorname{Vol} B\left(\frac{1}{2} m(Y)^{1/2}\right)}{\operatorname{Vol} P} = \frac{\operatorname{Vol} B\left(\frac{1}{2} m(Y)^{1/2}\right)}{\det Y^{1/2}}\,.$$

The *lattice packing problem* is to find the (volume normalised) lattice L with highest density Δ. Note that by (2.65) maximising Δ is equal to maximising $m(\cdot)$, i.e., the lattice packing problem can be reformulated as finding the lattice with the longest shortest vector. In case there is a unique solution to the lattice packing problem in dimension n, we will denote it by L_n (with Gram matrix Y_n),i.e., L_n is the unique solution (up to euclidean isometries fixing 0) to the problem

$$m(Y_n) \geqslant m(Y) \quad \text{for all } Y \in \mathcal{SP}_n / \operatorname{GL}(n;\mathbb{Z}) \tag{2.66}$$

with equality if and only if $[Y] = [Y_n]$.

(iii) The lattice packing problem has been solved in dimensions $n = 1, 2, \dots, 8$ and $n = 24$. The lattices introduced in *(i)* are precisely the solutions in their respective dimension $2, 4, 8, 24$. In particular, each of the lattices introduced in (i) is the unique global maximum of $m(\cdot)$ in $\mathcal{SP}_n / \operatorname{GL}(n;\mathbb{Z})$. For the hexagonal lattice A_2 and the checkerboard lattice D_4 this fact can be found in [**CS99**, Ch. 1 § 1.4], for the lattice E_8 in [**CS99**, Ch. 4 § 8.1]. That Λ_{24} is the unique lattice in dimension 24 maximising Δ was proved in [**CK09**]. We remark that all known solutions of the lattice packing problem have large isometry groups (see [**CS99**]).

DEFINITION 2.40. We call a $2n$-dimensional lattice L *symplectic* if it admits a Gram matrix of the form

$$Y = \begin{pmatrix} W & 0 \\ 0 & W^{-1} \end{pmatrix} \left[\begin{pmatrix} \mathrm{Id} & -X \\ 0 & \mathrm{Id} \end{pmatrix} \right] = \begin{pmatrix} \mathrm{Id} & 0 \\ -X & \mathrm{Id} \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & -X \\ 0 & \mathrm{Id} \end{pmatrix}, \tag{2.67}$$

where $W \in \mathcal{P}_n$ and $X \in M(n;\mathbb{R})$ is symmetric.

REMARK 2.41.

(i) Note that the matrices on the right hand side of (2.67) are all symplectic. Indeed

$$\begin{pmatrix} \mathrm{Id} & -X \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ -X & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} X & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ -X & \mathrm{Id} \end{pmatrix} = J\,,$$

and

$$\begin{pmatrix} W & 0 \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & W^{-1} \end{pmatrix} = \begin{pmatrix} 0 & W \\ -W^{-1} & 0 \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & W^{-1} \end{pmatrix} = J\,.$$

Hence, if a lattice L is symplectic it admits a Gram matrix $Y \in \mathcal{P}^*_{2n}$. On the other hand, every $Y \in \mathcal{P}^*_{2n}$ has partial Iwasawa coordinates as in (2.67), i.e., can be put into the form on the right hand side of (2.67) (see [**Ter88**, Chapter V, Lemma 2]). Hence, a lattice L is symplectic if and only if it admits a Gram matrix $Y \in \mathcal{P}^*_{2n}$.

We call a lattice L *isodual* if there is a euclidean isometry σ fixing 0 such that $\sigma(L) = L^*$. A symplectic lattice is necessarily isodual. In fact, if L is symplectic with symplectic Gram matrix Y, then

$${}^{\mathrm{t}}YJY = YJY = J\,,$$

so that

$$Y^{-1} = {}^{\mathrm{t}}JYJ = Y[J]\,,$$

i.e., Y^{-1}, which is the Gram matrix of L^*, can be obtained from Y through a (symplectic) change of basis (via $J \in \mathrm{Sp}(2n;\mathbb{Z})$).

(ii) It was shown in [**BS94**, Appendix 2] that, among others, D_4, E_8 and Λ_{24} are symplectic lattices. A symplectic Gram matrix for D_4 is given by

$$Y_4 = \begin{pmatrix} \mathrm{Id} & 0 \\ -X & \mathrm{Id} \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & -X \\ 0 & \mathrm{Id} \end{pmatrix}$$

with

$$X = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}, \qquad W = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{2.68}$$

An easy calculation shows that Y_4 is exactly the Gram matrix given in (2.63).

Furthermore, for $2n = 8, 24$ we let $Y_{2n} \in \mathcal{P}^*_{2n}$ be a Gram matrix for L_{2n}, i.e., Y_8 is a symplectic Gram matrix for E_8 and Y_{24} is a symplectic Gram matrix for Λ_{24}. Note that every two-dimensional lattice with volume one is symplectic since $\mathrm{Sp}(2;\mathbb{R}) = \mathrm{SL}(2;\mathbb{R})$ and thus $\mathcal{P}^*_2 = \mathcal{SP}_2$. In particular, the hexagonal lattice is symplectic. A symplectic Gram matrix Y_2 for A_2 is given in (2.62).

THEOREM 2.42 ([**OPS88b**, Corollary 1(b)]). *The height of flat tori $\zeta'(\cdot, 0)$ on $\mathcal{SP}_2/\,\mathrm{GL}(2;\mathbb{Z})$ attains a strict global minimum at Y_2, i.e., is globally minimised at the flat torus corresponding to the hexagonal lattice.*

THEOREM 2.43 (**[Ran53]**,**[Cas63]**,**[Cas59]**,**[Dia64]**,**[Enn64a]**).
For all $s > 0$ and $[Y] \in \mathcal{SP}_2 / \operatorname{GL}(2;\mathbb{Z})$ one has

$$\zeta_{Ep}(Y,s) - \zeta_{Ep}(Y_2,s) \geqslant 0\,, \tag{2.69}$$

with equality if and only if $[Y] = [Y_2]$.

REMARK 2.44.

(i) It follows from isoduality of the hexagonal lattice and (2.58) that (2.69) is also valid if one replaces Epstein's ζ-function by the spectral ζ-function of flat tori.

(ii) Note that Theorem 2.43 did not exclude the point $s = 2/2 = 1$ from its statement, although it is a pole of Epstein's ζ-function. In fact, (2.55) shows that the poles on the left hand side of (2.69) cancel. Thus, we get a statement about the finite part of Epstein's ζ-function at $s = 1$. Using (2.56) and (2.59) one can show that this statement is equivalent to Theorem 2.42.

THEOREM 2.45 (**[SS06]**, Theorem 1]). *For $n = 4, 8$ and 24 and $s > 0$, $\zeta_{Ep}(\cdot, s)$ has a strict local minimum at Y_n, as does the height of flat tori $\zeta'(\cdot, 0)$.*

REMARK 2.46. As in the case of the hexagonal lattice above, it follows from isoduality of L_n with $n \in \{4, 8, 24\}$ and (2.58) that we can replace Epstein's ζ-function with the spectral ζ-function of flat tori in the last theorem.

CONJECTURE 2.47 (P. Sarnak). The height of flat tori $\zeta'(\cdot, 0)$ on the moduli space $\mathcal{SP}_n / \operatorname{GL}(n;\mathbb{Z})$ of n-dimensional flat tori with volume one attains a global minimum at the torus corresponding to a lattice with the longest minimal vector.

REMARK 2.48. Theorem 2.43, Theorem 2.45 and work by Ennola **[Enn64b]** in dimension 3 raise the question if the analog of Sarnak's conjecture is true for Epstein's ζ-function at positive s, i.e., if

$$\zeta_{Ep}(Y,s) - \zeta_{Ep}(Y_n,s) \geqslant 0 \quad \text{for all } s > 0\,, \tag{2.70}$$

with equality if and only if $[Y] = [Y_n] \in \mathcal{SP}_n / \operatorname{GL}(n;\mathbb{Z})$, where Y_n is a Gram matrix for a solution L_n of the lattice packing problem. In fact, this was conjectured by R. A. Rankin (see **[Chi97]**).

An observation by Sarnak and Strömbergsson (see **[SS06]**, page 117) shows that (2.70) is in general false. A necessary criterion for (2.70) to hold is that the vector lengths of L_n are exactly the same as those of L_n^*: $\#\{m \in \mathbb{Z}^n \mid Y_n[m] = y\} = \#\{m \in \mathbb{Z}^n \mid Y_n^{-1}[m] = y\}$ for all $y \geqslant 0$. For if we let $G(s) := \pi^{-s}\Gamma(s)\zeta_{Ep}(Y_n, s) - \pi^{-s}\Gamma(s)\zeta_{Ep}(Y_n^{-1}, s)$ then $G(s) = -G(n/2 - s)$ by (2.56). Hence, G is either identically zero or changes sign. If G is identically zero, then with formula (2.55) one can see that the vector lengths of L_n and L_n^* must be the the same. If G changes sign then (2.70) fails. In dimension 3, the unique solution of the lattice packing problem is the *face centered cubic lattice* $L_3 = fcc$ with dual the *body centered cubic lattice* $L_3^* = bcc$. For these, we have $m(L_3) > m(L_3^*)$, see **[CS99**, Ch. 4 § 6.3]. In particular, the vector lengths of L_3 and L_3^* are not the same. Hence, (2.70) fails if $n = 3$.

Note that above necessary criterion is satisfied by all isodual lattices. In particular, by D_4, E_8 and Λ_{24}.

3.2. Extremal Metrics of The Height of normalised Heisenberg Manifolds. Our first result concerns the case of normalised manifolds of Heisenberg type.

THEOREM 2.49.

(i) Let $r \in \mathcal{D}_1 = \mathbb{N}$, $Y_{2,r} := r^{1/2} Y_2\left[\delta_r^{-1}\right]$ and $s \in (0,\infty)\backslash \operatorname{sing}(\dim H_1) = (0,\infty)\backslash\{3/2, 1/2\}$. Then on $\mathcal{SM}_1^{r,HT}$ both $[\mathbf{m}] \mapsto \zeta((\Gamma^r\backslash H_1, \mathbf{m}), s)$ and $[\mathbf{m}] \mapsto \zeta'((\Gamma^r\backslash H_1, \mathbf{m}), 0)$ have a strict global minimum at $[\mathbf{m}] = [(Y_{2,r}, r^{-1})]$.

(ii) Let $2n \in \{4, 8, 24\}$, $s \in (0,\infty)\backslash \operatorname{sing}(\dim H_n)$ and denote $\mathbb{1} = (1, 1, \dots, 1) \in \mathbb{N}^n$. Then on the moduli space $\mathcal{SM}_n^{\mathbb{1},HT}$ the spectral ζ-function $[\mathbf{m}] \mapsto \zeta((\Gamma^{\mathbb{1}}\backslash H_n, \mathbf{m}), s)$ and the height $[\mathbf{m}] \mapsto \zeta'((\Gamma^{\mathbb{1}}\backslash H_n, \mathbf{m}), 0)$ have a strict local minimum at $[\mathbf{m}] = [(Y_n, 1)]$.

PROOF. Let $n \in \mathbb{N}$, $r \in \mathcal{D}_n$ and $\mathbf{m} = (h, g) \in \mathcal{SM}_n^r$. We split the ζ-function and height into the respective sums

$$\zeta((\Gamma^r\backslash H_n, (h,g)), s) = \zeta_B((\Gamma^r\backslash H_n, (h,g)), s) + \zeta_F((\Gamma^r\backslash H_n, (h,g)), s),$$
$$\zeta'((\Gamma^r\backslash H_n, (h,g)), 0) = \zeta_B'((\Gamma^r\backslash H_n, (h,g)), 0) + \zeta_F'((\Gamma^r\backslash H_n, (h,g)), 0),$$

from Theorem 2.5 and Corollary 2.6. Recall that $h[\delta_r]$ is the Gram matrix of $\Gamma_{\mathfrak{n}}^r \subset (\mathfrak{n}, h)$. By Observation 2.38 we have

$$\Gamma(s)\zeta_B((\Gamma^r\backslash H_n, (h,g)), s) = (4\pi^2)^{-s}\Gamma(s)\zeta_{Ep}\left(h^{-1}\left[\delta_r^{-1}\right], s\right) - \frac{\operatorname{Vol} T_{\mathfrak{n},h}}{(4\pi)^n} \cdot \frac{1}{s-n},$$
$$\begin{aligned}\zeta_B'((\Gamma^r\backslash H_n, (h,g)), 0) &= 2\ln 2\pi + \zeta_{Ep}'\left(h^{-1}\left[\delta_r^{-1}\right], 0\right) + \frac{\operatorname{Vol} T_{\mathfrak{n},h}}{(4\pi)^n \cdot n} \\ &= \zeta'(T_{\mathfrak{n},h}, 0) + \frac{\operatorname{Vol} T_{\mathfrak{n},h}}{(4\pi)^n \cdot n}.\end{aligned}$$

On $\mathcal{SM}_n^{r,HT}$ the function $[\mathbf{m}] \mapsto \zeta_F((\Gamma^r\backslash H_n, \mathbf{m}), s)$ is constant (see Remark 2.8) and we have $g = |\Gamma^r|^{-2/(n+1)}$ and $h \in \mathcal{P}_{2n}^*(g^{1/2} \cdot \mathrm{Id})$ for all $\mathbf{m} = (h, g) \in \mathcal{SM}_n^{r,HT}$ (see Remark 1.71). It follows that on $\mathcal{SM}_n^{r,HT}$ the functions $[(h,g)] \mapsto \zeta((\Gamma^r\backslash H_n, (h,g)), s)$ and $[(h,g)] \mapsto \zeta'((\Gamma^r\backslash H_n, (h,g)), 0)$ have a strict global/local minimum if and only if $\mathcal{P}_{2n}^*(g^{1/2})/\Pi_r \ni [h] \mapsto \zeta_{Ep}\left(h^{-1}\left[\delta_r^{-1}\right], s\right)$ and $\mathcal{P}_{2n}^*(g^{1/2})/\Pi_r \ni [h] \mapsto \zeta'\left(T_{\mathfrak{n},h}, 0\right)$, respectively, have a strict global/local minimum.

By Remark 2.41 the lattices $L_4 = D_4$, $L_8 = E_8$ and $L_{24} = \Lambda_{24}$ are symplectic and thus, in particular, isodual. Note that $\delta_{\mathbb{1}} = \mathrm{Id}$. It follows from Theorem 2.45 and the above that

$$\mathcal{SM}_n^{\mathbb{1},HT} \ni [(h,1)] \mapsto \zeta((\Gamma^{\mathbb{1}}\backslash H_n,(h,1)),s) \quad \text{and}$$
$$\mathcal{SM}_n^{\mathbb{1},HT} \ni [(h,1)] \mapsto \zeta'((\Gamma^{\mathbb{1}}\backslash H_n,(h,1)),0)$$

have a strict local minimum at $[(h,1)] = [(Y_n,1)]$ for $2n \in \{4,8,24\}$. This proves *(ii)*.

To prove *(i)* let us first see why $[(Y_{2,r},r^{-1})] \in \mathcal{SM}_1^{r,HT}$: Since $\mathrm{Sp}(2;\mathbb{R}) = \mathrm{SL}(2;\mathbb{R})$ we have $r^{-1/2}\mathcal{SP}_2 = r^{-1/2}\mathcal{P}_2^* = \mathcal{P}_2^*(r^{-1/2}\cdot\mathrm{Id})$. From this and $\det Y_{2,r} = r\det\delta_r^{-2}\det Y_2 = r^{-1}\cdot\det Y_2 = r^{-1}\cdot 1$ it follows that $Y_{2,r} \in \mathcal{P}_2^*(r^{-1/2}\cdot\mathrm{Id})$, and hence $[(Y_{2,r},r^{-1})] \in \mathcal{SM}_1^{r,HT}$.

Further, we have

$$\zeta_{Ep}\left(Y_{2,r}^{-1}\left[\delta_r^{-1}\right],s\right) = \zeta_{Ep}\left(r^{-1/2}Y_2^{-1}\left[\delta_r\right]\left[\delta_r^{-1}\right],s\right) = r^{s/2}\zeta_{Ep}\left(Y_2^{-1},s\right) = r^{s/2}\zeta_{Ep}\left(Y_2,s\right),$$

where the last equality holds since L_2 is symplectic and thus, in particular, isodual, and

$$\begin{aligned}\zeta'(T_{\mathfrak{n},Y_{2,r}},0) = 2\ln 2\pi + \zeta'_{Ep}\left(Y_{2,r}^{-1}\left[\delta_r^{-1}\right],0\right) = 2\ln 2\pi + \tfrac{1}{2}\ln r\cdot\zeta_{Ep}\left(Y_2^{-1},0\right) + \\ + \zeta'_{Ep}\left(Y_2^{-1},0\right) = -\tfrac{1}{2}\ln r + \zeta'((\mathbb{Z}^2\backslash\mathbb{R}^2,Y_2),0)\,.\end{aligned}$$

The claim now follows from Theorems 2.43 and 2.42. □

We now return to general normalised Heisenberg manifolds.

THEOREM 2.50. *Let $n \in \mathbb{N}$, $r \in \mathcal{D}_n$, $s \in (0,\infty)\backslash\operatorname{sing}(\dim H_n)$ and $[\mathbf{m}_0] = [(h_0,g_0)] \in \mathcal{SM}_n^r$ be a (strict) local/global minimum/maximum for the ζ-function $[\mathbf{m}] \mapsto \zeta((\Gamma^r\backslash H_n,\mathbf{m}),s)$ or the height $[\mathbf{m}] \mapsto \zeta'((\Gamma^r\backslash H_n,\mathbf{m}),0)$ on $\mathcal{SM}_n^r$ or $\mathcal{SM}_{n,C}^r$. Then $[h_0]$ is a (strict) local/global minimum/maximum for*

$$\mathcal{P}_{2n}^*(h_0)/\Pi_r \ni [Y] \mapsto \zeta_{Ep}\left(Y^{-1}\left[\delta_r^{-1}\right],s\right) \quad \textit{or}$$
$$\mathcal{P}_{2n}^*(h_0)/\Pi_r \ni [Y] \mapsto \zeta'((\mathbb{Z}^{2n}\backslash\mathbb{R}^{2n},Y[\delta_r]),0)\,, \quad \textit{respectively.}$$

Recall from Observation 2.38 that $\zeta_{Ep}\left(Y^{-1}\left[\delta_r^{-1}\right],s\right) = (4\pi^2)^s\zeta((\mathbb{Z}^{2n}\backslash\mathbb{R}^{2n},Y[\delta_r]),s)$ and $\zeta'((\mathbb{Z}^{2n}\backslash\mathbb{R}^{2n},Y[\delta_r]),0) = 2\ln 2\pi + \zeta'_{Ep}\left(Y^{-1}\left[\delta_r^{-1}\right],0\right)$.

PROOF. Let us first mention why the above maps are well-defined. By definition (see (1.29)) we have $\Pi_r = \delta_r\cdot\mathrm{GL}(2n;\mathbb{Z})\cdot\delta_r^{-1}\cap\widetilde{Sp}(2n;\mathbb{Z})$. Hence, the map

$$\begin{aligned}\mathcal{P}_{2n}^*(h_0)/\Pi_r &\to \mathcal{P}_{2n}/\,\mathrm{GL}(2n;\mathbb{Z})\\ [Y] &\mapsto [Y^{-1}[\delta_r^{-1}]]\end{aligned}$$

is well-defined. By Remark 2.35, ζ_{Ep} is well-defined on $\mathcal{P}_{2n}/\,\mathrm{GL}(2n;\mathbb{Z})$.

We abbreviate “local/global minimum/maximum” to “glocal mimax”. Also, our argument involves only $\zeta_H((\Gamma^r\backslash H_n,\cdot),s)$ and $\zeta_{Ep}(\cdot,s)$. The respective statements about the

s-derivatives of these functions are obtained completely analogously; see the formula for ζ' in the proof of Theorem 2.49.

If $[(h_0, g_0)] \in \mathcal{SM}_n^r$ is a glocal mimax for $[\mathbf{m}] \mapsto \zeta((\Gamma^r\backslash H_n, \mathbf{m}), s)$ on $\mathcal{SM}_n^r$ then it is still a glocal mimax of $[\mathbf{m}] \mapsto \zeta((\Gamma^r\backslash H_n, \mathbf{m}), s)$ if we restrict the domain to

$$D := \{[(h, g_0)] \mid h \in \mathcal{P}_{2n}^*(h_0)\} \ .$$

By Proposition 1.49, Proposition 1.48 and the definition of D we have

$$c_n^{(h,g_0)} = g_0^{1/2} d_n(h) = g_0^{1/2} d_h(h_0) = c_n^{(h_0,g_0)}$$

for all $[(h, g_0)] \in D$, where $c_n^{\mathbf{m}}$ was defined in Definition 1.3*(iii)*. It follows from Proposition 1.12*(b)* that the sectional curvature satisfies $K_\sigma^{(h,g_0)} \leqslant C$ for all $[(h, g_0)] \in D$, i.e., $D \subset \mathcal{SM}_{n,C}^r$. Hence, the above statement is also true if one replaces $\mathcal{SM}_n^r$ by $\mathcal{SM}_{n,C}^r$.

Now we split $\zeta((\Gamma^r\backslash H_n, \mathbf{m}), s)$ into the sum

$$\zeta((\Gamma^r\backslash H_n, \mathbf{m}), s) = \zeta_B((\Gamma^r\backslash H_n, \mathbf{m}), s) + \zeta_F((\Gamma^r\backslash H_n, \mathbf{m}), s)$$

from Theorem 2.5. By Remark 2.8 the function $[\mathbf{m}] \mapsto \zeta_F((\Gamma^r\backslash H_n, \mathbf{m}), s)$ is constant on D. Furthermore, by Observation 2.38 the function $[(h, g)] \mapsto \zeta_B((\Gamma^r\backslash H_n, (h, g)), s)$ is given by

$$\Gamma(s)\zeta_B((\Gamma^r\backslash H_n, (h, g)), s) = (4\pi^2)^{-s}\Gamma(s)\zeta_{Ep}\left(h^{-1}\left[\delta_r^{-1}\right], s\right) - \frac{\det(4\pi h^{-1}\left[\delta_r^{-1}\right])^{-1/2}}{s-n} \ .$$

The statement now follows from $\det h = \det h_0$ for all $h \in \mathcal{P}_{2n}^*(h_0)$. □

REMARK 2.51. In light of Sarnak's Conjecture (see Conjecture 2.47), the last theorem suggests that natural places to look for global/local minima are those normalised Heisenberg manifolds $(\Gamma^r\backslash H_n, (h, g))$ whose base torus $T_{\mathfrak{n},h} = (\Gamma_{\mathfrak{n}}^r\backslash\mathbb{R}^{2n}, h)$ is isometric to $L_{2n}\backslash\mathbb{R}^{2n}$, where L_{2n} is as in 2.39*(ii)*. However, one has to be careful: While the isometry class of the torus $T_{\mathfrak{n},h}$, or equivalently of the Gram matrix $h[\delta_r]$, is invariant under the action of $G_r = \delta_r \operatorname{GL}(2n; \mathbb{Z})\delta_r^{-1}$ on $\mathcal{P}_{2n} \ni h$, the isometry class of h as part of the metric of the normalised Heisenberg manifold $(\Gamma^r\backslash H_n, (h, g))$ is invariant only under the action of $\Pi_r = \widetilde{\operatorname{Sp}}(2n; \mathbb{R}) \cap \delta_r \operatorname{GL}(2n; \mathbb{Z})\delta_r^{-1}$. This means that, e.g., natural candidates for minima of $[(h, g)] \mapsto \zeta'((\Gamma^{(1,1)}\backslash H_2, (h, g), 0)$ on $\mathcal{SM}_2^{(1,1)}$ are not only metrics of the form $(g_0^{-1/4} Y_4, g_0)$, where $g_0 \in (0, \infty)$ and Y_4 is the symplectic Gram matrix for the checkerboard lattice D_4 from Remark 2.41*(ii)*, but also all metrics of the form $(g_0^{-1/4}\, {}^{\mathrm{t}}G Y_4 G, g_0)$ with $G \in U$, where U is a complete set of representatives of $\Pi_{(1,1)}\backslash G_{(1,1)} = \widetilde{\operatorname{Sp}}(4; \mathbb{Z})\backslash\operatorname{GL}(4; \mathbb{Z})$.

THEOREM 2.52. *Let $s \in (0, \infty)\backslash\{3/2, 1/2\}$ and $r \in \mathcal{D}_1 = \mathbb{N}$. Let $[(h_0, g_0)]$ be a global minimum of one of the following three functions:*

(i) $\mathcal{SM}_1^r \ni [\mathbf{m}] \mapsto \zeta'((\Gamma^r\backslash H_n, \mathbf{m}), 0)$,

(ii) $\mathcal{SM}_1^r \ni [\mathbf{m}] \mapsto \zeta((\Gamma^r\backslash H_n, \mathbf{m}), s)$ *if* $s \in (0, 1/2)$,

(iii) $\mathcal{SM}_{1,C}^r \ni [\mathbf{m}] \mapsto \zeta((\Gamma^r\backslash H_n, \mathbf{m}), s)$ *if* $s \in (1/2, \infty)\backslash\{3/2\}$.

The existence of global minima in the cases (i) and (ii) is asserted by Theorem 2.11(iii) and (iv), respectively. The existence of a global minimum in case (iii) is guaranteed by Theorem 2.30(iv).

Then $[h_0] = \left[g_0^{-1/2} Y_2[\delta_r^{-1}]\right]$*, i.e., the base torus* $T_{\mathfrak{n},h_0} \simeq (\mathbb{Z}^2\backslash\mathbb{R}^2, g_0^{-1/2}Y_2)$ *corresponds to a rescaled hexagonal lattice.*

PROOF. We only prove case *(i)*. The other two cases can be obtained with minor changes. By Theorem 2.50 we know that $[h_0]$ must be a global minimum for

$$\mathcal{P}_2^*(h_0)/\Pi_r \ni [Y] \mapsto \zeta'\left((\mathbb{Z}^2\backslash\mathbb{R}^2, Y[\delta_r]), 0\right) = 2\ln 2\pi + \zeta'_{Ep}\left(Y^{-1}\left[\delta_r^{-1}\right], 0\right). \tag{2.71}$$

By rescaling (see Remark 2.37) we see that $\left[r \cdot g_0^{1/2} h_0\right]$ is a global minimum for

$$\mathcal{P}_2^*\left(r \cdot g_0^{1/2} h_0\right)/\Pi_r \ni [Y] \mapsto \zeta'\left((\mathbb{Z}^2\backslash\mathbb{R}^2, Y[\delta_r]), 0\right).$$

Note that $r \cdot g_0^{1/2} h_0 \in \mathrm{SL}(2;\mathbb{R}) = \mathrm{Sp}(2;\mathbb{R})$, hence $\mathcal{P}_2^*\left(r \cdot g_0^{1/2} h_0\right) = \mathcal{SP}_2 = \{Y \in \mathcal{P}_2 \mid \det Y = 1\}$. By Theorem 2.42 we necessarily have

$$\left[r^{-1} \cdot \left(r \cdot g_0^{1/2} h_0\right)[\delta_r]\right] = \left[g_0^{1/2} h_0 [\delta_r]\right] = [Y_2] \text{ in } \mathcal{SP}_2/\,\mathrm{GL}(2;\mathbb{Z}),$$

which is equivalent to

$$[h_0] = \left[g_0^{-1/2} Y_2 \left[\delta_r^{-1}\right]\right] \text{ in } \mathcal{P}_2/\Pi_r.$$

□

In the context of Theorem 2.52, note that we do not know for which $g_0 \in (0, \infty)$ the minimal value of $\zeta'(\cdot, 0)$ or $\zeta(\cdot, s)$ is achieved, although its existence is guaranteed as stated in the theorem.

As we have mentioned above, natural places to look for minima of $\zeta'(\cdot, 0)$ are those normalised Heisenberg manifolds, which have as base torus a version of $L_{2n}\backslash\mathbb{R}^{2n}$. On these last pages we present some calculations that suggest that there exists $g_0 \in (0, \infty)$ such that $\left[\left(g_0^{-1/4} Y_4, g_0\right)\right] \in \mathcal{SM}_2^{(1,1)}$ is a minimum for $\zeta'((\Gamma_{(1,1)}\backslash H_2, \cdot), 0)$. While we cannot prove that this metric is a minimum, we present these calculations in the hopes that they turn out useful to someone.

We already know that Y_4 is a local minimum for

$$\mathcal{SP}_4 \ni Y \mapsto \zeta'_B\left((\Gamma^{(1,1)}\backslash H_2, (Y, 1)), 0\right) = \zeta'\left((\mathbb{Z}^4\backslash\mathbb{R}^4, Y), 0\right) + \frac{\det Y}{(4\pi)^2 \cdot 2}$$

by (2.61) and Theorem 2.45, and we will show that Y_4 is a critical point for $\mathcal{SP}_4 \ni Y \mapsto \zeta'_F((\Gamma_{(1,1)}\backslash H_2, (Y, 1)), 0)$. Unfortunately, we are unable to prove that it is a (nonstrict) local minimum (recall that $Y \mapsto \zeta_F'((\Gamma_{(1,1)}\backslash H_2,(Y,1)),0)$ is constant on $\mathcal{P}_4^*(\mathrm{Id}) = \mathcal{P}_4^* \subset \mathcal{SP}_4$ by Remark 2.8).

First, we recall the formula for $\zeta_F'((\Gamma^{(1,1)}\backslash H_2,(h,1)),0)$ with $h \in \mathcal{P}_4$ from Corollary 2.6:

$$\zeta_F'((\Gamma^{(1,1)}\backslash H_2,(h,1)),0) = \tag{2.72}$$
$$\frac{1}{(4\pi)^2}\int_1^\infty \sum_{\lambda\in\mathbb{Z}\backslash\{0\}} s_{V,t}^{(h,1)}(\lambda)t^{-3}\,\mathrm{d}t + \frac{1}{(4\pi)^{5/2}}\int_0^1 \sum_{X\in\mathbb{Z}\backslash\{0\}} \sigma_X^{(h,1)}(t)t^{-7/2}\mathrm{d}t$$
$$+\frac{1}{(4\pi)^{5/2}}\left(\int_0^1\left(\sigma_0^{(h,1)}(t) - T_{+0}^2[\sigma_0^{(h,1)}](t)\right)t^{-7/2}\,\mathrm{d}t + \sum_{j=0}^2 \frac{a_j^{(h,1)}}{j-5/2}\right),$$

where $T_{+0}^2[\sigma_0^{(h,1)}](t) = \sum_{j=0}^2 a_j^{(h,1)}t^j$. Also, recall formulas (1.42) and (1.43):

$$s_{V,t}^{(h,1)} : \mathfrak{z} = \mathbb{R} \ni \lambda \mapsto e^{-4\pi^2\lambda^2 t}\prod_{j=1}^2 \frac{2\pi\lambda d_j(h)t}{\sinh\left(2\pi\lambda d_j(h)t\right)} \in \mathbb{R},$$
$$\sigma_X^{(h,1)} : [0,\infty) \ni t \mapsto \pi^{-1/2}\int_{\mathbb{R}} e^{-\frac{i}{\sqrt{t}}X\cdot\xi}e^{-\xi^2}\prod_{j=1}^2 \frac{\sqrt{t}d_j(h)\xi}{\sinh\left(\sqrt{t}d_j(h)\xi\right)}\mathrm{d}\xi \in \mathbb{R},$$

where we have used $c_j^{(h,1)} = d_j(h)$ (see Proposition 1.49).

We will show that $h = Y_4 \in \mathcal{P}_4$ is a critical point for each term in the above formula for ζ_F'. Recall the following from Chapter 1, Section 2. The tangent spaces of $\mathcal{SP}_4$ and $\mathcal{P}_4^*$ at the identity are given by

$$T_{\mathrm{Id}}\mathcal{SP}_4 = \mathfrak{q}_4 = \left\{X \in M(4;\mathbb{R}) \mid {}^{\mathrm{t}}X = X, \operatorname{tr} X = 0\right\},$$
$$T_{\mathrm{Id}}\mathcal{P}_4^* = \mathfrak{p}_4 = \mathfrak{q}_4 \cap \mathfrak{sp}(4;\mathbb{R}) = \left\{X \in M(4;\mathbb{R}) \mid {}^{\mathrm{t}}X = X, XJ+JX = 0\right\}.$$

The Riemannian metric on $\mathcal{SP}_4$ and $\mathcal{P}_4^*$ is invariant under the $\mathrm{SL}(4;\mathbb{R})$- and $\mathrm{Sp}(4;\mathbb{R})$-action, respectively. On $\mathfrak{q}_4 \subset \mathfrak{p}_4$ it is given by $(X,Y) \mapsto \operatorname{tr}(XY)$. The matrix exponential map $\exp : \mathfrak{q}_4 \to \mathcal{SP}_4$ is a diffeomorphism with $\exp(\mathfrak{p}_4) = \mathcal{P}_4^*$. Moreover, exp maps the lines through 0 to geodesics through Id. In particular, $\mathcal{P}_4^*$ is a totally geodesic submanifold of $\mathcal{SP}_4$.

Denote by $E_{i,j} \in M(4;\mathbb{R})$ the elementary matrix with a 1 in the i-th row and j-th column and 0 everyhwere else.

PROPOSITION 2.53. *Let*

$$H_1 := \frac{1}{\sqrt{2}}\left(E_{1,1} - E_{3,3}\right), \qquad H_2 := \frac{1}{\sqrt{2}}\left(E_{2,2} - E_{4,4}\right),$$
$$G_{1,2} := \frac{1}{2}\left(E_{1,2} + E_{2,1} - E_{3,4} - E_{4,3}\right), \quad K_1 := \frac{1}{\sqrt{2}}\left(E_{1,3} + E_{3,1}\right),$$

$$K_2 := \frac{1}{\sqrt{2}}\left(E_{2,4}+E_{4,2}\right), \qquad L_{1,2} := \frac{1}{2}\left(E_{1,4}+E_{2,3}+E_{3,2}+E_{4,1}\right),$$
$$M := \frac{1}{2}\left(E_{1,1}-E_{2,2}+E_{3,3}-E_{4,4}\right), \quad N := \frac{1}{2}\left(E_{1,4}-E_{2,3}-E_{3,2}+E_{4,1}\right),$$
$$P := \frac{1}{2}\left(E_{2,1}+E_{1,2}+E_{3,4}+E_{4,3}\right).$$

With respect to the innder product $(X,Y) \mapsto \mathrm{tr}(XY)$, the tuple $(H_1, H_2, G_{1,2}, K_1, K_2, L_{1,2})$ constitutes an orthonormal basis of $\mathfrak{p}_4$ and (M,N,P) is an orthonormal basis of the orthogonal complement of $\mathfrak{p}_4$ in $\mathfrak{q}_4$.

PROOF. We only remark that $E_{i,j}E_{k,l} = \delta_{j,k}E_{i,l}$ and leave the simple but tedious calculations needed to prove the claim to the disbelieving reader. □

DEFINITION AND REMARKS 2.54. We define two maps $\varphi : \mathbb{R}^3 \to \mathfrak{q}_4$ and $\Phi : \mathbb{R}^3 \to \mathcal{P}_4$ by

$$\varphi : \mathbb{R}^3 \ni x = (u,v,w) \mapsto u\cdot M + v\cdot N + w\cdot P = \begin{pmatrix} u/2 & w/2 & 0 & v/2 \\ w/2 & -u/2 & -v/2 & 0 \\ 0 & -v/2 & u/2 & w/2 \\ v/2 & 0 & w/2 & -u/2 \end{pmatrix},$$

$$\Phi : \mathbb{R}^3 \ni x = (u,v,w) \mapsto Y_4^{1/2}\sum_{j=0}^{\infty}\frac{\varphi(x)^j}{j!}Y_4^{1/2} = (\exp\varphi(x))\left[Y_4^{1/2}\right].$$

By Proposition 2.53 the image of φ is the orthogonal complement of $\mathfrak{p}_4$ in $\mathfrak{q}_4$. Since Y_4 is symplectic, so is any power of it (see, e.g., [**dG06**, Chapter 2]), in particular $Y_4^{1/2}$. Moreover, $Y_4^{1/2}$ acts isometrically on $\mathcal{P}_4$ and thus leaves $\mathcal{P}_4^*$ invariant. It follows that the image of the differential $\mathrm{d}\Phi_0$ is precisely the orthogonal complement of $T_{Y_4}\mathcal{P}_4^* \subset T_{Y_4}\mathcal{SP}_4$.

Note that $\varphi(x)^2 = \frac{1}{4}\|x\|^2 \cdot \mathrm{Id}$. It follows that

$$\exp\varphi(x) = \sum_{j=0}^{\infty}\frac{\varphi(x)^{2j}}{(2j)!} + \sum_{j=0}^{\infty}\frac{\varphi(x)^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\|x\|\right)^{2j}}{(2j)!}\cdot\mathrm{Id} + \sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\|x\|\right)^{2j}}{(2j+1)!}\varphi(x)$$
$$= \cosh\left(\tfrac{1}{2}\|x\|\right)\cdot\mathrm{Id} + \sinh\left(\tfrac{1}{2}\|x\|\right)\frac{\varphi(x)}{\frac{1}{2}\|x\|}. \tag{2.73}$$

Hence,

$$\Phi(x) = \cosh\left(\tfrac{1}{2}\|x\|\right)\cdot Y_4 + \frac{\sinh\left(\frac{1}{2}\|x\|\right)}{\frac{1}{2}\|x\|}\cdot Y_4^{1/2}\varphi(x)Y_4^{1/2}.$$

PROPOSITION 2.55. *We have $d_1(\Phi(x)) = e^{-1/2\|x\|}$ and $d_2(\Phi(x)) = e^{1/2\|x\|}$.*

PROOF. By definition the numbers $\pm i d_j(\Phi(x))$, $j=1,2$, are the eigenvalues of

$$\Phi(x)^{-1}J = Y_4^{-1/2}\exp(-\varphi(x))Y_4^{-1/2}J \sim \exp\varphi(-x)Y_4^{-1/2}JY_4^{-1/2} = \exp\varphi(-x)J,$$

where the last equality holds since $Y_4^{1/2}$ is symplectic, see [**dG06**, Chapter 2]. An easy calculation shows that J commutes with $\varphi(-x) = -\varphi(x)$ for all $x \in \mathbb{R}^3$. Hence, J also commutes with $\exp\varphi(-x)$ for all $x \in \mathbb{R}^3$. The $+i$ eigenspace E_i of J is spanned by $v_1 := e_1 + ie_3$ and $v_2 := e_2 + ie_4$ where (e_1, e_2, e_3, e_4) is the standard basis of $\mathbb{C}^4$. Since $\exp\varphi(-x)$ and J commute, $\exp\varphi(-x)$ leaves E_i invariant and we have

$$\exp\varphi(-x)v_1 = \left(\cosh\left(\tfrac{1}{2}\|x\|\right) - \frac{\sinh\left(\frac{1}{2}\|x\|\right)}{\frac{1}{2}\|x\|}\frac{x_1}{2}\right)v_1 - \frac{\sinh\left(\frac{1}{2}\|x\|\right)}{\frac{1}{2}\|x\|}\frac{x_3 - ix_2}{2}v_2\,,$$
$$\exp\varphi(-x)v_2 = \left(\cosh\left(\tfrac{1}{2}\|x\|\right) + \frac{\sinh\left(\frac{1}{2}\|x\|\right)}{\frac{1}{2}\|x\|}\frac{x_1}{2}\right)v_2 - \frac{\sinh\left(\frac{1}{2}\|x\|\right)}{\frac{1}{2}\|x\|}\frac{x_3 + ix_2}{2}v_1\,.$$

This means that in the basis (v_1, v_2) of E_i the linear map $(\exp\varphi(-x))\big|_{E_i}$ has the matrix representation

$$A := \begin{pmatrix} \cosh\left(\frac{1}{2}\|x\|\right) - \frac{\sinh\left(\frac{1}{2}\|x\|\right)}{\frac{1}{2}\|x\|}\frac{x_1}{2} & -\frac{\sinh\left(\frac{1}{2}\|x\|\right)}{\frac{1}{2}\|x\|}\frac{x_3+ix_2}{2} \\ -\frac{\sinh\left(\frac{1}{2}\|x\|\right)}{\frac{1}{2}\|x\|}\frac{x_3-ix_2}{2} & \cosh\left(\frac{1}{2}\|x\|\right) + \frac{\sinh\left(\frac{1}{2}\|x\|\right)}{\frac{1}{2}\|x\|}\frac{x_1}{2} \end{pmatrix}.$$

The matrix A has eigenvalues

$$\begin{aligned}\lambda_1 &= \tfrac{1}{2}\operatorname{tr}A + \sqrt{\tfrac{(\operatorname{tr}A)^2}{4} - \det A} = \cosh\left(\tfrac{1}{2}\|x\|\right) + \sqrt{\cosh^2\left(\tfrac{1}{2}\|x\|\right) - 1} \\ &= \cosh\left(\tfrac{1}{2}\|x\|\right) + \sinh\left(\tfrac{1}{2}\|x\|\right) = e^{1/2\|x\|}\end{aligned}$$

and

$$\begin{aligned}\lambda_2 &= \tfrac{1}{2}\operatorname{tr}A - \sqrt{\tfrac{(\operatorname{tr}A)^2}{4} - \det A} = \cosh\left(\tfrac{1}{2}\|x\|\right) - \sqrt{\cosh^2\left(\tfrac{1}{2}\|x\|\right) - 1} \\ &= \cosh\left(\tfrac{1}{2}\|x\|\right) - \sinh\left(\tfrac{1}{2}\|x\|\right) = e^{-1/2\|x\|}\,,\end{aligned}$$

from which $d_1(\Phi(x)) = e^{-1/2\|x\|}$ and $d_2(\Phi(x)) = e^{1/2\|x\|}$ follows. $\square$

LEMMA 2.56. *Let $T^2_{+0}\left[\sigma_0^{(\Phi(x),1)}\right](t) = a_0 + a_1(x)t + a_2(x)t^2$. Then*

$$a_0(x) \equiv 1\,,\quad a_1(x) = -\tfrac{1}{6}\cosh\|x\|\,,\quad a_2(x) = \tfrac{1}{120}\left(7\cosh^2\|x\| - 1\right).$$

In particular, $T^2_{+0}\left[\sigma_0^{(\Phi(x),1)}\right](t)$ has a local maximum at $x = 0$ for every $t \in (0, 1]$.

PROOF. Recall that $c_j^{(h,1)} = d_j(h)$. Hence, by Lemma 1.90 we have $a_0 = 1$, $a_1(x) = -\frac{1}{12}\left(d_1(\Phi(x))^2 + d_2(\Phi(x))^2\right)$ and

$$a_2(x) = \tfrac{1}{480}\left(7\left(d_1(\Phi(x))^2 + d_2(\Phi(x))^2\right)^2 - 4d_1(\Phi(x))^2 d_2(\Phi(x))^2\right).$$

By the last proposition we have $d_1(\Phi(x)) = e^{-1/2\|x\|}$ and $d_2(\Phi(x)) = e^{1/2\|x\|}$, from which the claimed formulas follow. By **[OLBC10**, 4.33.2] we have $\cosh\|x\| = 1 + \frac{1}{2}\|x\|^2 + O(\|x\|^4)$. Hence $\cosh^2\|x\| = 1 + \|x\|^2 + O(\|x\|^4)$. It follows that

$$T^2_{+0}\left[\sigma_0^{(\Phi(x),1)}\right](t) = \left(1 - \frac{t}{6} + \frac{t^2}{20}\right) + \left(-\frac{t}{12} + \frac{7\cdot t^2}{120}\right)\cdot\|x\|^2 + O(\|x\|^4) \text{ for all } t \geqslant 0\,.$$

Since the solutions of $-t/12 + 7/120\cdot t^2 = 0$ are $t = 10/7 > 1$ and $t = 0$ and since the coefficient of t^2 is positive, $-t/12 + 7/120\cdot t^2 < 0$ for all $t \in (0,1]$. In particular, $T^2_{+0}\left[\sigma_0^{(\Phi(x),1)}\right](t)$ has a local maximum at $x = 0$ for every $t \in (0,1]$. ☐

COROLLARY 2.57. *The function*

$$x \mapsto \sum_{j=0}^{2} \frac{a_j(x)}{-5/2 + j}$$

has a local maximum at $x = 0$.

PROOF. By the last lemma we have

$$\sum_{j=0}^{2} \frac{a_j(x)}{-5/2 + j} = -\frac{2}{5} + \frac{1}{9}\cosh\|x\| - \frac{1}{60}\left(7\cosh^2\|x\| - 1\right)\,.$$

Using the expansions $\cosh\|x\| = 1 + \frac{1}{2}\|x\|^2 + O(\|x\|^4)$ and $\cosh^2\|x\| = 1 + \|x\|^2 + O(\|x\|^4)$ we obtain

$$\sum_{j=0}^{2} \frac{a_j(x)}{-5/2 + j} = -\frac{7}{18} - \frac{11}{180}\|x\|^2 + O(\|x\|^4)\,,$$

from which the claim follows. ☐

PROPOSITION 2.58. *For any $\tau \in \mathbb{R}\backslash\{0\}$ the function*

$$\mathbb{R}^3 \ni x \mapsto \frac{\tau d_1(\Phi(x))}{\sinh(\tau d_1(\Phi(x)))}\frac{\tau d_2(\Phi(x))}{\sinh(\tau d_2(\Phi(x)))}$$

has a local maximum at $x = 0$. Consequently, the functions $\sigma_0^{(\Phi(x),1)}(t)$ and $\int_1^\infty \sum_{\lambda\in\mathbb{Z}\backslash\{0\}} s_{V,t}^{(\Phi(x),1)}(\lambda)t^{-3}\mathrm{d}t$ have local maxima in $x = 0$.

PROOF. We abbreviate $d_j(\Phi(x))$ to $d_j(x)$ and start by transforming the function. For this, let $\tau \in \mathbb{R}\backslash\{0\}$ be arbitrary. Then we have

$$\begin{aligned}\frac{\tau d_1(x)}{\sinh(\tau d_1(x))}\frac{\tau d_2(x)}{\sinh(\tau d_2(x))} &= \frac{4\tau^2}{\left(e^{\tau d_1(x)} - e^{-\tau d_1(x)}\right)\left(e^{\tau d_2(x)} - e^{-\tau d_2(x)}\right)} \\ &= \frac{2\tau^2}{\cosh(\tau(d_1(x) + d_2(x))) - \cosh(\tau(d_1(x) - d_2(x)))}\end{aligned}$$

$$= \frac{2\tau^2}{\cosh(2\tau\cosh\|x\|) - \cosh(2\tau\sinh\|x\|)} =: \frac{2\tau^2}{F(x)}\,.$$

The function $f : \mathbb{R} \ni t \mapsto \cosh(2\tau\cosh t) - \cosh(2\tau\sinh t)$ is even, smooth in $t = 0$ and clearly $F(x) = f(\|x\|)$. Hence, F is differentiable in $x = 0$ and has a local minimum in $x = 0$ if and only if f has a local minimum in $t = 0$. Since f is even, f' is odd and $t = 0$ a critical value of f. We compute $f''(0)$:

$$\begin{aligned}
f''(0) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (2\tau\sinh(2\tau\cosh t)\sinh t - 2\tau\sinh(2\tau\sinh t)\cosh t)\\
&= \Big(4\tau^2\cosh(2\tau\cosh t)\sinh^2 t + 2\tau\cosh t\sinh(2\tau\cosh t)\\
&\quad - 4\tau^2\cosh(2\tau\sinh t)\cosh^2 t - 2\tau\sinh t\sinh(2\tau\sinh t)\Big)\Big|_{t=0}\\
&= 2\tau\sinh(2\tau) - 4\tau^2 = 2\tau\sum_{j=0}^{\infty}\frac{(2\tau)^{2j+1}}{(2j+1)!} - 4\tau^2 = \sum_{j=0}^{\infty}\frac{(4\tau^2)^{j+1}}{(2j+1)!} - 4\tau^2 > 0\,.
\end{aligned}$$

So the function F has indeed a local minimum in $x = 0$. This in turn means that $\frac{2\tau^2}{F(x)}$ has a local maximum in $x = 0$. Since $\tau \neq 0$ was arbitrary, any summation or integration over values of τ leaves $x = 0$ a local maximum. □

COROLLARY 2.59. *The point $x = 0$ is a critical point of $\int_0^1 \sum_{X\in\mathbb{Z}\setminus\{0\}} \sigma_X^{(\Phi(x),1)}(t)t^{-7/2}\mathrm{d}t$.*

PROOF. By formula (1.43) we have

$$\begin{aligned}
&\int_0^1 \sum_{X\in\mathbb{Z}\setminus\{0\}} \sigma_X^{(\Phi(x),1)}(t)t^{-7/2}\mathrm{d}t \\
&= \pi^{-1/2}\int_0^1 \sum_{X\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} e^{-\frac{i}{\sqrt{t}}X\cdot\xi} e^{-\xi^2} F(t,\xi,x)\mathrm{d}\xi\, t^{-7/2}\mathrm{d}t\,.
\end{aligned} \tag{2.74}$$

where

$$F(t,\xi,x) := \prod_{j=1}^{2} \frac{\sqrt{t}d_j(\Phi(x))\xi}{\sinh\left(\sqrt{t}d_j(\Phi(x))\xi\right)}\,.$$

It follows from Proposition 2.58 that $x = 0$ is a critical point of $\mathbb{R}^3 \ni x \mapsto F(t,\xi,x)$ for all $t \geqslant 0$ and $\xi \in \mathbb{R}$, and hence also of (2.74). □

By (2.72) we have proved that $h = Y_4$ is a critical point for $\mathcal{SP}_4 \ni h \mapsto \zeta_F'((\Gamma_{(1,1)}\backslash H_2,(h,1)),0)$. Since we already knew that $h = Y_4$ is a critical point of $\mathcal{SP}_4 \ni h \mapsto \zeta_B'((\Gamma_{(1,1)}\backslash H_2,(h,1)),0)$, we have proved the following theorem.

THEOREM 2.60. *The metric $h = Y_4$ is a critical point of the function*

$$\mathcal{SP}_4 \ni h \mapsto \zeta_F'((\Gamma^{(1,1)}\backslash H_2, (h,1)), 0) \in \mathbb{R}\,.$$

In particular, $h = Y_4$ is a critical point of the function

$$\begin{aligned}\mathcal{SP}_4 \ni h \mapsto\ & \zeta'((\Gamma^{(1,1)}\backslash H_2, (h,1)), 0) \\ &= \zeta_B'((\Gamma^{(1,1)}\backslash H_2, (h,1)), 0) + \zeta_F'((\Gamma^{(1,1)}\backslash H_2, (h,1)), 0) \in \mathbb{R}\,.\end{aligned}$$

Bibliography

[BCY92] T. P. Branson, S.-Y. A. Chang, and P. C. Yang. Estimates and extremals for zeta function determinants on four-manifolds. *Commun. Math. Phys.*, 149(2):241–262, 1992.

[BGM71] M. Berger, P. Gauduchon, and E. Mazet. *Le spectre d'une variété riemannienne. (The spectrum of a Riemannian manifold).* Lecture Notes in Mathematics. 194. Berlin-Heidelberg-New York: Springer-Verlag , 1971.

[Bha96] R. Bhatia. *Matrix analysis.* Springer New York, 1996.

[Bla98] H. Blanchard. *Autour Des Nilvariétés Isospectrales*. PhD thesis, l'Institut Fourier de l'Université Joseph Fourier (Grenoble I), 1998.

[BØ91] T. P. Branson and B. Ørsted. Explicit functional determinants in four dimensions. *Proc. Am. Math. Soc.*, 113(3):669–682, 1991.

[Bra93] T. P. Branson. *The functional determinant.* Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, 1993.

[BS94] P. Buser and P. Sarnak. On the period matrix of a Riemann surface of large genus (with an appendix by J. H. Conway and N. J. A. Sloane). *Invent. Math.*, 117(1):27–56, 1994.

[Cas59] J.W.S. Cassels. On a problem of Rankin about the Epstein zeta-function. *Proc. Glasg. Math. Assoc.*, 4:73–80, 1959.

[Cas63] J.W.S. Cassels. Corrigendum to the paper "On a problem of Rankin about the Epstein zeta- function". *Proc. Glasg. Math. Assoc.*, 6:116, 1963.

[CD02] G. Crandall and J. Dodziuk. Integral structures on H-type Lie algebras. *J. Lie Theory*, 12(1):69–79, 2002.

[CG90] L. J. Corwin and F. P. Greenleaf. *Representations of nilpotent Lie groups and their applications. Part 1: Basic theory and examples.* Cambridge University Press, 1990.

[Chi97] P. Chiu. Height of flat tori. *Proc. Am. Math. Soc.*, 125(3):723–730, 1997.

[CK09] H. Cohn and A. Kumar. Optimality and uniqueness of the Leech lattice among lattices. *Ann. Math. (2)*, 170(3):1003–1050, 2009.

[CS99] J.H. Conway and N.J.A. Sloane. *Sphere packings, lattices and groups. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.* Springer New York, 3rd ed. edition, 1999.

[CY95] S.-Y. A. Chang and P. C. Yang. Extremal metrics of zeta function determinants on 4-manifolds. *Ann. Math. (2)*, 142(1):171–212, 1995.

[dG06] M. de Gosson. *Symplectic geometry and quantum mechanics*, volume 166 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2006.

[Dia64] P.H. Diananda. Notes on two lemmas concerning the Epstein zeta-function. *Proc. Glasg. Math. Assoc.*, 6:202–204, 1964.

[Ebe94] P. Eberlein. Geometry of 2-step nilpotent groups with a left invariant metric. *Ann. Sci. Éc. Norm. Supér. (4)*, 27(5):611–660, 1994.

[Els05] J. Elstrodt. *Maß- und Integrationstheorie*. Springer-Lehrbuch. Springer-Verlag, Berlin, fourth edition, 2005.

[EMOT54] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Tables of integral transforms. Vol. I.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954.

[Enn64a] V. Ennola. A lemma about the Epstein zeta-function. *Proc. Glasg. Math. Assoc.*, 6:198–201, 1964.

[Enn64b] V. Ennola. On a problem about the Epstein zeta-function. *Proc. Camb. Philos. Soc.*, 60:855–875, 1964.

[Eps03] P. Epstein. Zur Theorie allgemeiner Zetafunctionen. *Math. Ann.*, 56:615–644, 1903.

[Fas07] G. E. Fasshauer. *Meshfree approximation methods with Matlab.* Hackensack, NJ: World Scientific, 2007.

[Fd03] K. Furutani and S. de Gosson. Determinant of Laplacians on Heisenberg manifolds. *J. Geom. Phys.*, 48(2-3):438–479, 2003.

[GL87] P.M. Gruber and C.G. Lekkerkerker. *Geometry of numbers. 2nd ed.* North Holland, 1987.

[GM00] R. Gornet and M. B. Mast. The length spectrum of Riemannian two-step nilmanifolds. *Ann. Sci. École Norm. Sup. (4)*, 33(2):181–209, 2000.

[Gra14] L. Grafakos. *Classical Fourier analysis.* Springer New York, 3rd ed. edition, 2014.

[Gur97] M. J. Gursky. Uniqueness of the functional determinant. *Commun. Math. Phys.*, 189(3):655–665, 1997.

[GW86] C. S. Gordon and E. N. Wilson. The spectrum of the Laplacian on Riemannian Heisenberg manifolds. *Mich. Math. J.*, 33:253–271, 1986.

[GW88] C. S. Gordon and E. N. Wilson. Isometry groups of Riemannian solvmanifolds. *Trans. Am. Math. Soc.*, 307(1):245–269, 1988.

[Kap81] A. Kaplan. Riemannian nilmanifolds attached to Clifford modules. *Geom. Dedicata*, 11:127–136, 1981.

[Lee02] J. M. Lee. *Introduction to smooth manifolds.* Springer New York, 2002.

[LM89] H.B. jun. Lawson and M.-L. Michelsohn. *Spin geometry.* Princeton Mathematical Series. 38. Princeton, NJ: Princeton University Press, 1989.

[Mø12] N. M. Møller. Extremal metrics for spectral functions of Dirac operators in even and odd dimensions. *Adv. Math.*, 229(2):1001–1046, 2012.

[MØ14] N. M. Møller and B. Ørsted. Rigidity of conformal functionals on spheres. *Int. Math. Res. Not.*, 2014(22):6302–6339, 2014.

[MP49] S. Minakshisundaram and A. Pleijel. Some properties of the Eigenfunctions of the Laplace-operator on Riemannian manifolds. *Can. J. Math.*, 1:242–256, 1949.

[Oki01] K. Okikiolu. Critical metrics for the determinant of the Laplacian in odd dimensions. *Ann. of Math. (2)*, 153(2):471–531, 2001.

[OLBC10] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST handbook of mathematical functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010.

[O'N83] B. O'Neill. *Semi-Riemannian geometry. With applications to relativity.* Academic Press, 1983.

[Ono82] E. Onofri. On the positivity of the effective action in a theory of random surfaces. *Commun. Math. Phys.*, 86:321–326, 1982.

[OPS88a] B. Osgood, R. Phillips, and P. Sarnak. Compact isospectral sets of surfaces. *J. Funct. Anal.*, 80(1):212–234, 1988.

[OPS88b] B. Osgood, R. Phillips, and P. Sarnak. Extremals of determinants of Laplacians. *J. Funct. Anal.*, 80(1):148–211, 1988.

[OPS89] B. Osgood, R. Phillips, and P. Sarnak. Moduli space, heights and isospectral sets of plane domains. *Ann. Math. (2)*, 129(2):293–362, 1989.

[Pes93] H. Pesce. Calcul du spectre d'une nilvariété de rang deux et applications. *Trans. Amer. Math. Soc.*, 339(1):433–461, 1993.

[Pes94] H. Pesce. Une formule de Poisson pour les variétés de Heisenberg. *Duke Math. J.*, 73(1):79–95, 1994.

[PS61] R.S. Palais and T.E. Stewart. Torus bundles over a torus. *Proc. Am. Math. Soc.*, 12:26–29, 1961.

[Ran53] R.A. Rankin. A minimum problem for the Epstein zeta-function. *Proc. Glasg. Math. Assoc.*, 1:149–158, 1953.

[Ric94] K. Richardson. Critical points of the determinant of the Laplace operator. *J. Funct. Anal.*, 122(1):52–83, 1994.

[See67] R.T. Seeley. Complex powers of an elliptic operator. Proc. Sympos. Pure Math. 10, 288-307 (1967)., 1967.

[SS06] P. Sarnak and A. Strömbergsson. Minima of Epstein's zeta function and heights of flat tori. *Invent. Math.*, 165(1):115–151, 2006.

[SSV12] R. L. Schilling, R. Song, and Z. Vondraček. *Bernstein functions. Theory and applications.* Berlin: de Gruyter, 2nd revised and extended edition, 2012.

[Ter85] A. Terras. *Harmonic analysis on symmetric spaces and applications. I.* Springer New York, 1985.

[Ter88] A. Terras. *Harmonic analysis on symmetric spaces and applications. II.* Springer New York, 1988.

[Wol64] J.A. Wolf. Curvature in nilpotent Lie groups. *Proc. Am. Math. Soc.*, 15:271–274, 1964.

[Zed65] M. Zedek. Continuity and location of zeros of linear combinations of polynomials. *Proc. Am. Math. Soc.*, 16:78–84, 1965.

Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014, angegebenen Hilfsmittel angefertigt habe.

Berlin, den 20.06.2017

Sebastian Boldt